4. Fourier series

Recall that a function $f(x)$ is <u>L-periodic</u> if $f(x+L) = f(x)$.

<u>Big idea</u>: Every piecewise continuous 2L-periodic function can be written using sine & cosine waves: $\cos\left(\frac{2\pi n}{L}x\right)$ $n \geq 0$ and $\sin\left(\frac{2\pi n}{L}x\right)$, $n \geq 1$.

<u>Goal</u>: Figure out <u>how</u> to do this.

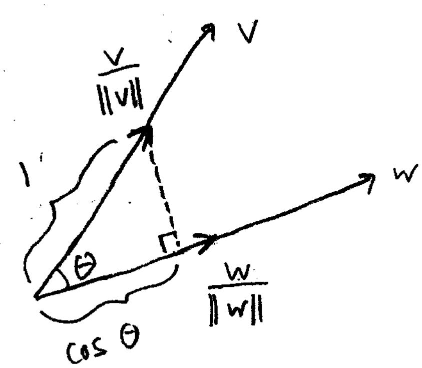
<u>Motivation</u>: Consider the vector space $\mathbb{R}^n$.

The dot product allows us to measure lengths and angles, and we can use this to project vectors onto other vectors.

* length: $\|v\| = \sqrt{v \cdot v}$

* angle: $\measuredangle(v, w) = \theta$, where $\cos\theta = \frac{v}{\|v\|} \cdot \frac{w}{\|w\|}$

$$= \frac{v \cdot w}{\|v\|\,\|w\|}$$



* projection: $\mathrm{proj}_w(v) = \left(\frac{v}{\|v\|} \cdot \frac{w}{\|w\|}\right) w = \frac{v \cdot w}{\|v\|\,\|w\|} w$

"projection of v onto the w-direction"

Note that if $\|e\| = 1$, then $\mathrm{proj}_e(v) = (v \cdot e)\,e$.

<u>Recall</u>: A set of vectors $\{v_1, \dots, v_n\}$ is <u>orthonormal</u> if $\langle v_i, v_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$.

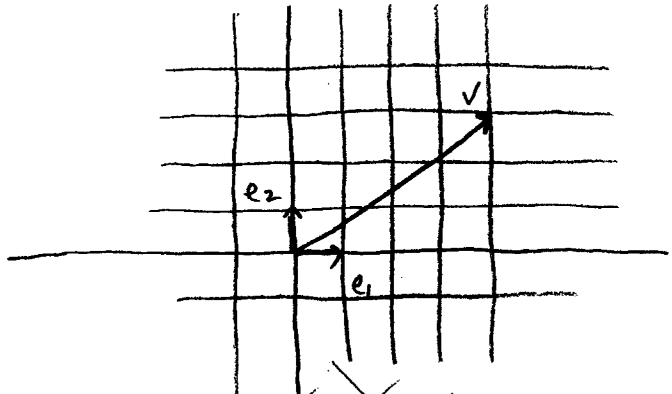
[2]

We can use projections to <u>decompose</u> a given vector v with respect to an orthonormal basis.

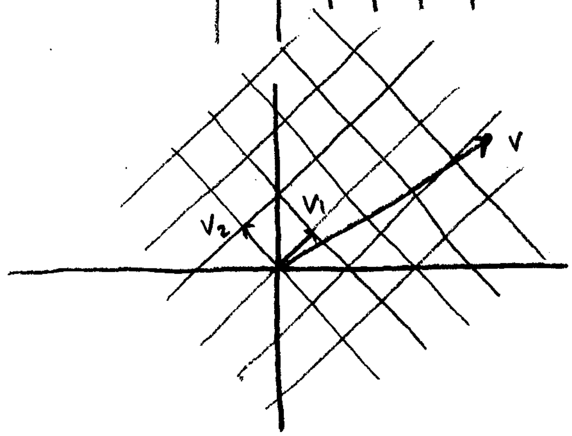
<u>Example</u>: Let $v = (4,3) \in \mathbb{R}^2$.

Let $e_1 = (1,0)$, $e_2 = (0,1)$

$v_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $v_2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



<u>Note</u>: Both $\{e_1, e_2\}$ and $\{v_1, v_2\}$ are orthonormal bases of $\mathbb{R}^2$.



$V = a_1 e_1 + a_2 e_2$

$= 4(1,0) + 3(0,1) = (4,3)$.

<u>Note</u>: $a_1 = v \cdot e_1 = 4$, $a_2 = v \cdot e_2 = 3$.

<u>OR</u>: $v = b_1 v_1 + b_2 v_2$, where $b_1 = v \cdot v_1 = (4,3) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{7\sqrt{2}}{2} \approx 4.95$

$b_2 = v \cdot v_2 = (4,3) \cdot \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} \approx -0.71$.

* In general, if $\{v_1, \dots, v_n\}$ is an <u>orthonormal basis</u> of V w.r.t. the inner product $\langle \,,\rangle$, then we can uniquely <u>decompose</u> any $v \in V$ in this basis by writing $v = a_1 v_1 + \dots + a_n v_n$, where $a_i = \langle v, v_i \rangle$.

Now, let $\boxed{V = Per_{2\pi}}$.

<u>Note</u>: Per_{2L} works too, but we'll mainly use $Per_{2\pi}$ because the math isn't as messy.

Fact: The set $\{\cos nx : n \geq 0\} \cup \{\sin nx : n \geq 1\}$ is a basis for $Per_{2\pi}$.

Define an inner product on $Per_{2\pi}$ as follows:

$$\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx.$$

Let's compute the magnitude of some functions (vectors):

$$\|\cos nx\|^2 = \langle \cos nx, \cos nx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} (\cos nx)^2\, dx = 1$$

$$\|\sin nx\|^2 = \langle \sin nx, \sin nx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} (\sin nx)^2\, dx = 1.$$

$$\|1\|^2 = \langle 1, 1 \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\, dx = 2. \quad \text{Thus, } \|1\| = \sqrt{2}.$$

This means that $\left\|\frac{1}{\sqrt{2}}\right\| = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\, dx\right)^{1/2} = 1.$

It can also be checked that:

$$\langle \cos nx, \cos mx \rangle = 0 \quad \text{if } n \neq m$$

$$\langle \sin nx, \sin mx \rangle = 0 \quad \text{if } n \neq m$$

$$\langle \cos nx, \sin mx \rangle = 0$$

$$\left\langle \frac{1}{\sqrt{2}}, \cos nx \right\rangle = \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} \cos nx\, dx = 0$$

$$\left\langle \frac{1}{\sqrt{2}}, \sin nx \right\rangle = \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} \sin nx\, dx = 0.$$

* Conclusion: $\mathcal{B}_{2\pi} := \left\{\frac{1}{\sqrt{2}}, \cos x, \cos 2x, \dots\right\} \cup \{\sin x, \sin 2x, \dots\}$ is an orthonormal basis of $Per_{2\pi}$!

This means that we can decompose any $f \in Per_{2\pi}$ by writing

$$f(x) = A_0 \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where $a_n = \langle f(x), \cos nx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad n \geq 1$

$b_n = \langle f(x), \sin nx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \qquad n \geq 1$

$A_0 = \langle f(x), \frac{1}{\sqrt{2}} \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f(x) \, dx = \frac{1}{\sqrt{2}}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \, dx \right).$

Convention: We usually write $\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx}$

because now, $a_0 = \sqrt{2} A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \, dx.$ This is the Fourier Series of f.

Advantage: We can now say that $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$

for all $n \geq 0$ (not just $n \geq 1$).

Remark: As mentioned before, we can any $2L$-periodic (not just 2π) function in a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 0$

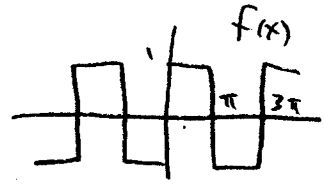
$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 1.$

Think of a_n as the "magnitude of $f(x)$ in the $\cos\left(\frac{n\pi x}{L}\right)$-direction"

and b_n as the "magnitude of $f(x)$ in the $\sin\left(\frac{n\pi x}{L}\right)$-direction."

Examples:

(1) Square wave: $f(x) = \begin{cases} 1 & 0 \le x \le \pi \\ -1 & -\pi \le x < 0 \end{cases}$



(extended to be 2π-periodic)

Find the Fourier series of $f(x)$

That is, write $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, and

find a_0, a_n, b_n.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1 \cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1 \cos nx\,dx$$

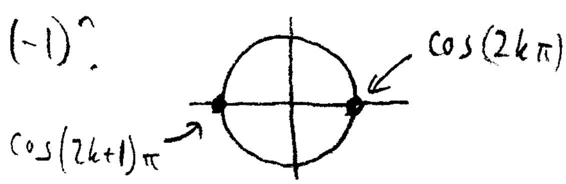
$$= -\frac{1}{n\pi}\sin nx\Big|_{-\pi}^{0} + \frac{1}{n\pi}\sin nx\Big|_{0}^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1 \sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1 \sin nx\,dx$$

$$= \frac{1}{n\pi}\cos nx\Big|_{-\pi}^{0} - \frac{1}{n\pi}\cos nx\Big|_{0}^{\pi} = \frac{1}{n\pi}(1-\cos n\pi) - \frac{1}{n\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n\pi}(1-\cos n\pi) = \boxed{\frac{2}{n\pi}\left(1-(-1)^n\right)}.$$

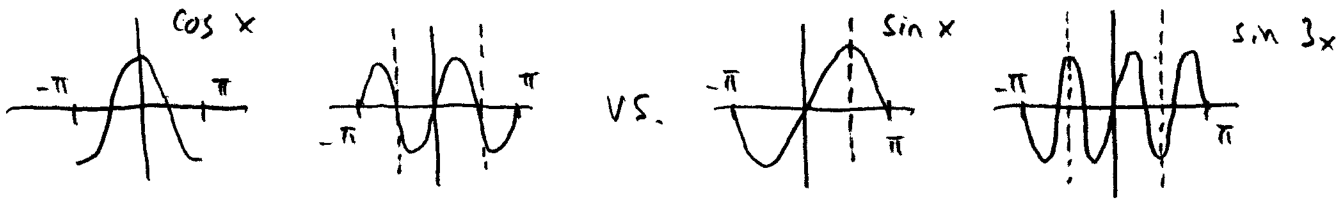
Note: $\cos n\pi = (-1)^n$.



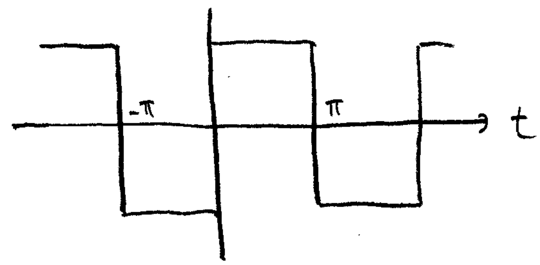
Therefore, $a_0 = 0$, $a_n = 0$, $b_n = \frac{2}{n\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$

i.e., $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(-1)^n\right) \sin nx = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \dots$

Remark: All cosine terms, and "even-index" sine terms are zero. (Why?)

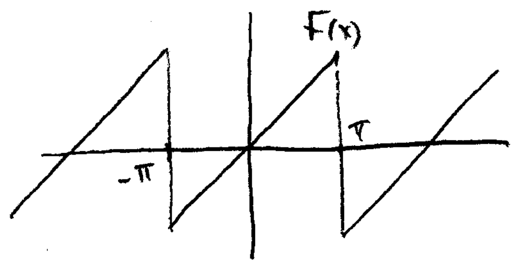


Look at the "symmetries" of $f(x)$:



This "looks like" a sine wave, and "more like" a $\sin x$, $\sin 3x$, etc. wave than a $\sin 2x$, $\sin 4x$ etc. wave.

(2) Sawtooth wave: $f(x) = x$ on $[-\pi, \pi]$, extended to be 2π-periodic.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \, dx = 0 \quad \text{(By symmetry; area under the curve)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx \qquad \text{let } u = x \quad v = \frac{1}{n}\sin nx \qquad du = dx \quad dv = \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n} x \overset{0}{\cancel{\sin nx}} \Big|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi} \sin nx \, dx \right]$$

$$= -\frac{1}{n\pi}\int_{-\pi}^{\pi} \sin nx \, dx = \frac{1}{n^2\pi}\cos nx \Big|_{-\pi}^{\pi} = \frac{1}{n^2\pi}\left[\cos(\pi n) - \cos(-\pi n)\right] = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \qquad \text{let } u = x \quad v = -\frac{1}{n}\cos nx$$
$$\qquad du = dx \quad dv = \sin nx \, dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{n} x \cos nx \Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nx \, dx \right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi}{n}\cos n\pi\right) - \left(\frac{\pi}{n}\cos n\pi\right) + \frac{1}{n^2}\overset{0}{\cancel{\sin nx}} \Big|_{-\pi}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi \right] = -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} = \begin{cases} -2/n & n \text{ even} \\ 2/n & n \text{ odd.} \end{cases}$$

Thus, $f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} = 2\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x +/- \cdots$

Think: How does this relate to music, sound waves, and harmonics?

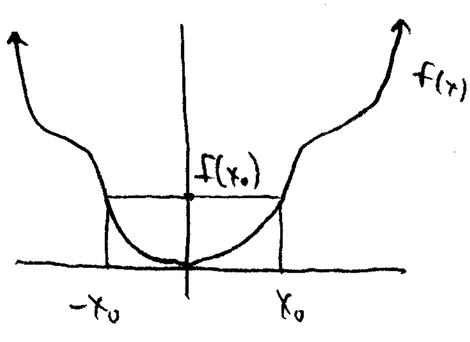
Exploiting symmetry:

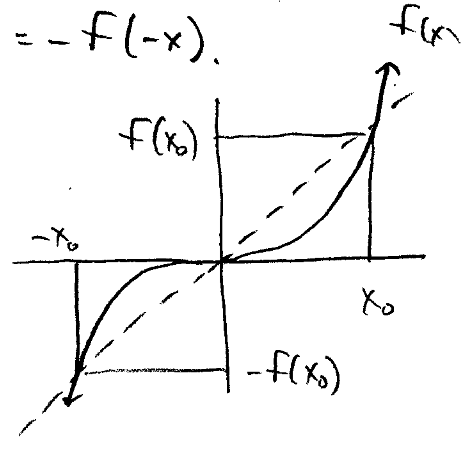
Question: Why are so many of the a_n's & b_n's zero?

Def: • $f(x)$ is an even function if $f(x) = f(-x)$.

• $f(x)$ is an odd function if $f(x) = -f(-x)$.

Graphically:





$f(x)$ even $\Leftrightarrow$ symmetric about y-axis. $\qquad$ $f(x)$ odd $\Leftrightarrow$ symmetric about origin

Why we care:

• If $f(x)$ is even, then $\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$

• If $f(x)$ is odd, then $\int_{-L}^{L} f(x)\,dx = 0$

Look at the area under the curve to see why!

<u>Basic facts</u>: • If f & g are even, then $f(x)\,g(x)$ is even.

• If f & g are odd, then $f(x)\,g(x)$ is even.

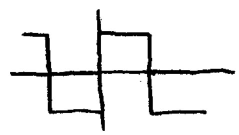
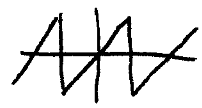
• If f is even & g is odd, then $f(x)\,g(x)$ is odd.

Examples:

<u>Even functions</u>: 8, x^2, $3x^6+x^2-5$, $|x|$, 

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} +/- \cdots = \frac{e^{ix}+e^{-ix}}{2}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \frac{e^{x}+e^{-x}}{2}$$

<u>Odd functions</u>: $2x$, $8x^3-5x$,  

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +/- \cdots = \frac{e^{ix}-e^{-ix}}{2}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \frac{e^{x}-e^{-x}}{2}.$$

<u>Neither</u>: x^2-3x+2, x^5+x^3+x+1, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

<u>Remarks</u>:

* $\cos x = \cosh ix$, $\quad i \sin x = \sinh ix$

* $e^x = \cosh x + \sinh x = \cos x + i \sin x$

* $\frac{d}{dx} \sinh x = \cosh x$ and $\frac{d}{dx} \cosh x = \sinh x$

* $\cosh 0 = \cos 0 = 1$, $\quad \sinh 0 = \sin 0 = 0$.

* The Taylor series of even functions contain only even terms, & Taylor series of odd functions contain only odd terms.

<u>Theorem:</u> Let $f(x)$ be an arbitrary function. Then $f(x)$ can be decomposed (uniquely) as $f(x) = f_{even}(x) + f_{odd}(x)$, where f_{even} is even and f_{odd} is odd.

<u>Proof:</u> (Existence): Put $f_{even}(x) = \dfrac{f(x)+f(-x)}{2}$, $f_{odd}(x) = \dfrac{f(x)-f(-x)}{2}$

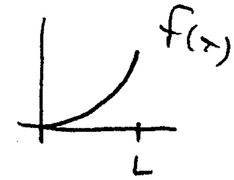
<u>Key point</u>:

- If $f(x)$ is <u>even</u>, then $f(x)\cos\left(\frac{n\pi x}{L}\right)$ is <u>even</u> $\Rightarrow a_n = \frac{4}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

  and $f(x)\sin\left(\frac{n\pi x}{L}\right)$ is odd $\Rightarrow b_n = 0$ (all n).
- If $f(x)$ is <u>odd</u>, then $f(x)\cos\left(\frac{n\pi x}{L}\right)$ is odd $\Rightarrow a_n = 0$ (all n)

  and $f(x)\sin\left(\frac{n\pi x}{L}\right)$ is even $\Rightarrow b_n = \frac{4}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$.

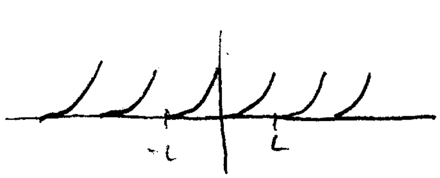
<u>Fourier Sine & Cosine series</u>:



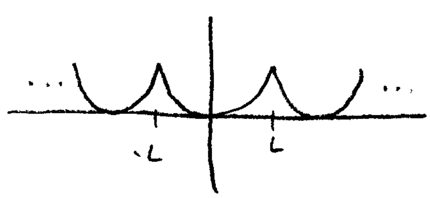
<u>Idea</u>: Consider a function defined on $[0, L]$.

Suppose we want / need to write $f(x)$ as a Fourier series.

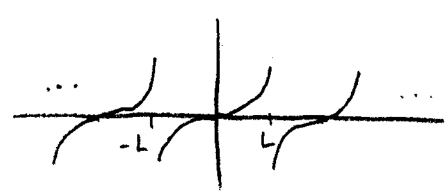
First, we need to <u>make</u> $f(x)$ periodic, by <u>extending</u> it.



The Fourier <u>series</u> extension



The <u>even</u> extension



The <u>odd</u> extension

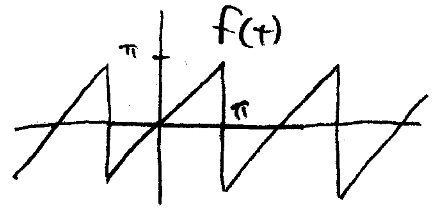
Def: The <u>Fourier cosine series</u> of $f(x)$ is the Fourier series of the even extension of $f(x)$

$$\begin{cases} a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n = 0 \end{cases}$$

Def: The <u>Fourier sine series</u> of $f(x)$ is the Fourier series of the odd extension of $f(x)$

$$\begin{cases} a_n = 0 \\ b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{cases}$$

<u>Motivating example</u>: Solve $x'' + w^2 x = f(t)$, where:



This models a simple harmonic oscillator with a driving force that's a sawtooth wave.

<u>Sol'n</u>: We know $x(t) = x_h(t) + x_p(t)$, where $x_h(t) = A\cos wt + B\sin wt$.

Assume that $x_p(t)$ has the form $x_p(t) = \sum_{n=1}^{\infty} b_n \sin nt$, and find the b_n's.

Plug back in: $\sum_{n=1}^{\infty} -n^2 b_n \sin nt + w^2 \sum_{n=1}^{\infty} b_n \sin nt = \underbrace{\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nt}_{\text{Recall Example 2}}$

$$\sum_{n=1}^{\infty} (w^2 - n^2) b_n \sin nt = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin t$$

$$\Rightarrow (w^2 - n^2) b_n = \frac{2}{n}(-1)^{n+1} \quad \Rightarrow \quad b_n = \frac{2(-1)^{n+1}}{n(w^2-n^2)}$$

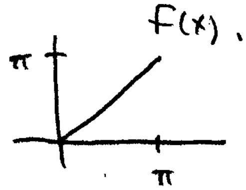
Thus, our general solution is $x(t) = x_h(t) + x_p(t)$

$$= A\cos wt + B\sin wt + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n(w^2-n^2)} \sin nt$$

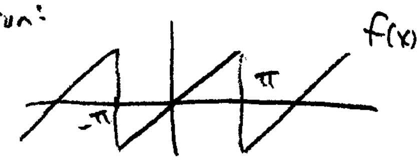
<u>Remark</u>: This is <u>much</u> easier than using Laplace transforms!

Examples (of Fourier sine & cosine series):

(3) Let $f(x) = x$ on $[0, \pi]$.



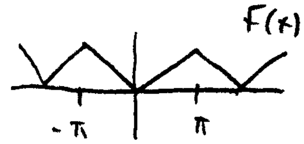
Fourier sine series: Odd extension:



This was Example 2, on p. 6-7.

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx.$$

Fourier cosine series: Even extension:



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \, dx = \frac{x^2}{\pi}\Big|_0^{\pi} = \pi.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx = \frac{2}{\pi}\left[\frac{x}{n}\cancelto{0}{\sin nx}\Big|_0^{\pi} - \int_0^{\pi} \frac{1}{n}\sin nx \, dx\right]$$

Let $u = x$ $\quad v = \frac{1}{n}\sin nx$

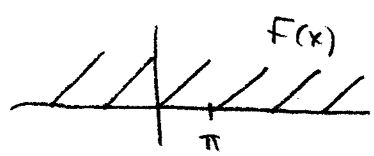
$du = dx$ $\quad dv = \cos nx \, dx$

$$= \frac{2}{\pi n^2}\cos nx\Big|_0^{\pi} = \frac{2}{n^2\pi}\left[\cos n\pi - 1\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & n \text{ even} \\ -\frac{4}{\pi n^2} & n \text{ odd} \end{cases}$$

Thus, $f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{\pi n^2}\cos nx = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25\pi}\cos 5x - \cdots$

Fourier series Fourier extension:



Note: $f(x)$ now has period $2L = \pi$ (not $2L = 2\pi$, as above). So $\boxed{L = \frac{\pi}{2}}$

It is not even nor odd, so there will be both a_n's and b_n's.

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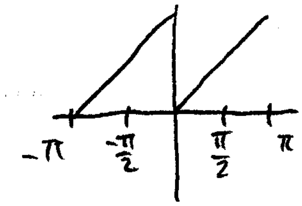
We'll have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2nx + b_n \sin 2nx$



$$a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos 2nx\,dx, \quad b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\sin 2nx\,dx$$

<u>Annoyance</u>: $f(x)$ isn't continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

<u>Work-around</u>: It isn't important that we integrate on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, but rather that we integrate on one <u>full period</u> of f. Thus, we'll do $[0,\pi]$ instead.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{x^2}{\pi}\Big|_0^{\pi} = \pi.$$

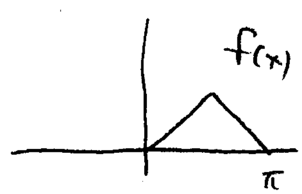
$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos 2nx\,dx = \frac{2}{\pi}\left(\frac{2nx\,\cancelto{0}{\sin(2nx)} + \cos(2nx)}{4n^2}\right)\Bigg|_0^{\pi} = \frac{\cos 2n\pi}{2\pi n^2}\Bigg|_0^{\pi} = 0.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin 2nx\,dx = \frac{2}{\pi}\left(\frac{\cancelto{0}{\sin(2nx)} - 2nx\cos(2nx)}{4n^2}\right)\Bigg|_0^{\pi} = \frac{-x\cos(2nx)}{\pi n}\Bigg|_0^{\pi} = -\frac{1}{n}$$

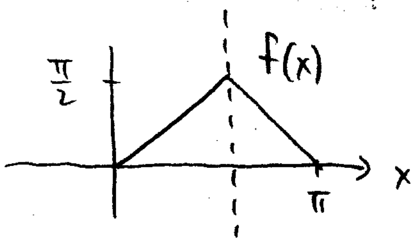
Thus, $f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{\sin 2nx}{n} = \frac{\pi}{2} - \sin 2x - \frac{1}{2}\sin 4x - \frac{1}{3}\sin 6x - \frac{1}{4}\sin 8x - \cdots$

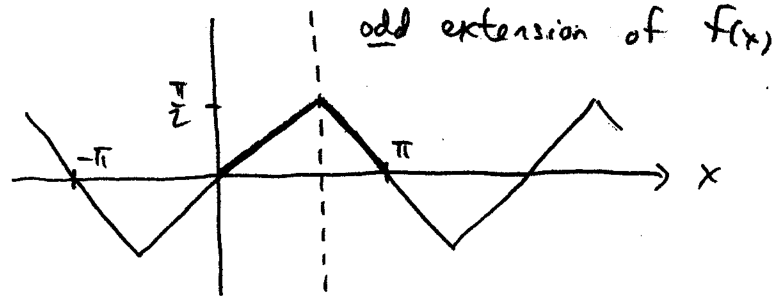
<u>Examples</u> (cont.)

(4) Let $f(x) = \begin{cases} x & 0 \le x < \pi/2 \\ \pi - x & \pi/2 \le x < \pi. \end{cases}$

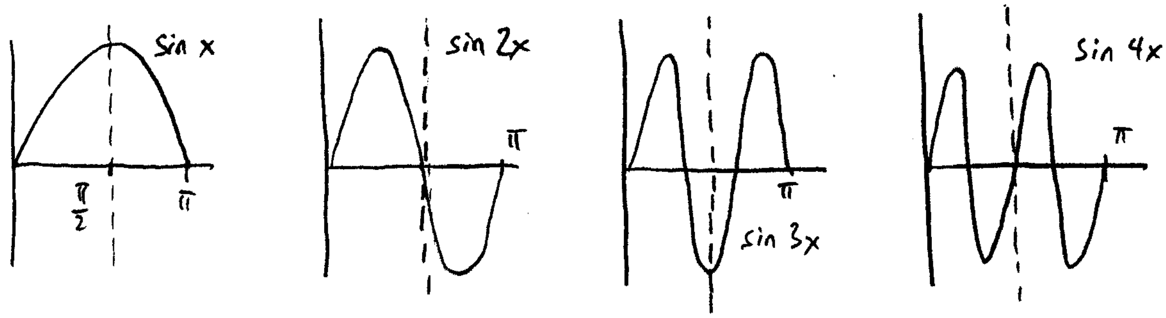


Compute the Fourier sine series of $f(x)$.





Observe the symmetry about the line $x = \pi/2$.



* $\sin nx$ has <u>odd</u> symmetry about the line $x = \pi/2$ if n is <u>even</u>.
* $\sin nx$ has <u>even</u> symmetry about the line $x = \pi/2$ if n is <u>odd</u>

<u>Conclusion</u>: If n is even, then $b_n = 0$.

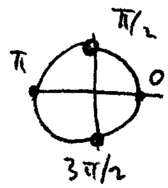
If n is <u>odd</u>, then $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{4}{\pi}\int_0^{\pi/2} f(x)\sin nx\,dx$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x\sin nx\,dx = \frac{4}{\pi}\left[\frac{x}{n}\cos nx\Big|_0^{\frac{\pi}{2}}\right] + \int_0^{\frac{\pi}{2}}\frac{1}{n}\cos nx\,dx$$

$$= \frac{4}{\pi}\left[-\frac{\pi}{2\pi}\overset{0}{\cancel{\cos}\left(\frac{n\pi}{2}\right)} - 0 + \frac{1}{n^2}\sin nx\Big|_0^{\frac{\pi}{2}}\right]$$

n odd $\Rightarrow \cos\left(\frac{n\pi}{2}\right) = 0$.

$$= \frac{4}{\pi}\left[\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]_0^{\frac{\pi}{2}}$$



$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n = 4k \\ 1 & n = 4k+1 \\ 0 & n = 4k+2 \\ -1 & n = 4k+3 \end{cases}$$

$$\text{Thus, } b_n = \begin{cases} 0 & n = 4k \\ 4/n^2\pi & n = 4k+1 \\ 0 & n = 4k+2 \\ -4/n^2\pi & n = 4k+3 \end{cases}$$

So, $F(x) = \frac{4}{\pi}\sin x - \frac{4}{9\pi}\sin 3x + \frac{4}{25\pi}\sin 5x - \frac{4}{49\pi}\sin 7x \; +/- \cdots$

Convergence of Fourier series

Consider a Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$.

This defines a $2L$-periodic function.

In fact, it is only necessary to define $f(x)$ on $[-L, L]$.

This is actually the "proper" way to define Fourier series, i.e.,

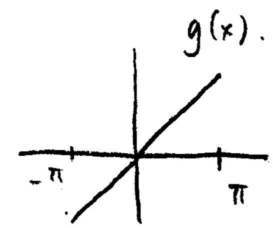
$\left\{ \frac{1}{\sqrt{2}}, \cos \frac{n\pi x}{L}, \sin \frac{n\pi x}{L} \right\}$ is an orthonormal basis for the space of piecewise functions $[-L, L] \longrightarrow \mathbb{R}$.

Given such a function $g(x)$, we can extend it to a periodic function $f: \mathbb{R} \to \mathbb{R}$. But it's not clear what $f(L)$ or $f(-L)$ will be.

Theorem: $f(L) = f(-L) = \frac{g(L) + g(-L)}{2}$, i.e, the Fourier series converges to the average value at a point of discontinuity.

The same holds for interior points of discontinuity.

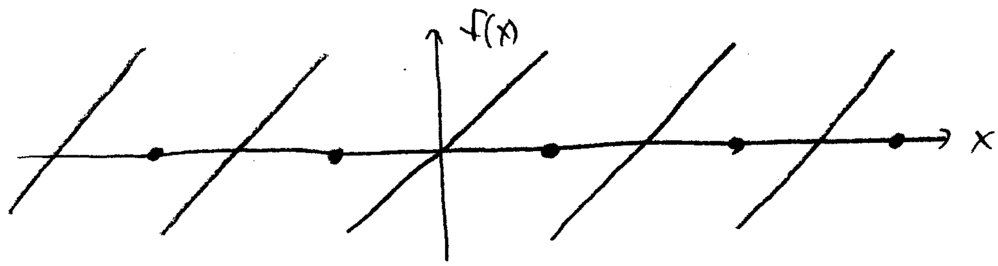
Example 2 (revisited): Let $g(x) = x$ on $[-\pi, \pi]$



The Fourier series of $g(x)$ is the function

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin nx$$

By the above theorem, $f(\pi) = f(-\pi) = 0$. So $f(x)$ actually looks like:



<u>Remark</u>: Given a Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$,

* The <u>y-intercept</u> of f is $f(0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$.
* The <u>average value</u> of f is $\frac{a_0}{2}$. (why?)

Complex Fourier series

<u>Recall</u>: $\mathcal{B}_1 = \left\{ \frac{1}{\sqrt{2}}, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) : n, m \in \mathbb{N} \right\}$ is a <u>basis</u> for $Per_{2L}(\mathbb{R})$, (or better: The piecewise functions $[-L, L] \longrightarrow \mathbb{R}$.) This basis is <u>orthonormal</u> w.r.t. the inner product $\langle f, g \rangle := \frac{1}{L}\int_{-L}^{L} f(x)\, g(x)\, dx$.

<u>Theorem:</u> The set $\{ e^{\frac{in\pi x}{L}} : n \in \mathbb{Z} \}$ is a basis for $Per_{2L}(\mathbb{C})$, (i.e, the piecewise functions $[-L, L] \longrightarrow \mathbb{C}$.) Moreover, this basis is <u>orthonormal</u> w.r.t. the inner product $\langle f, g \rangle := \frac{1}{2L}\int_{-L}^{L} f(x)\, \overline{g(x)}\, dx$

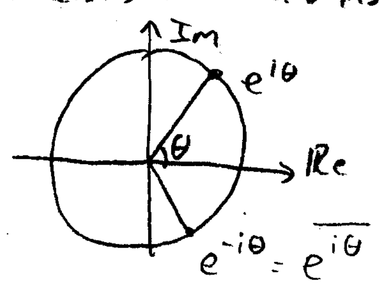
Therefore, if $f(x)$ is 2L-periodic, we can write it as

$$\boxed{\begin{gathered} f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{\frac{in\pi x}{L}} + c_{-n} e^{\frac{-in\pi x}{L}} \right) \\ c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx, \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{-in\pi x}{L}}\, dx \end{gathered}}$$

<u>Remark</u>: $\overline{a+bi} = a - bi$, the complex conjugate. (Reflection across the Re. axis.

Since $e^{i\theta}$ is on the unit circle at θ radians,

$\overline{e^{i\theta}} = e^{-i\theta}$.



Remark: It is easy to go between the real and complex versions of a Fourier series.

Recall: $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$, $\quad \sin x = \dfrac{e^{ix}-e^{-ix}}{2i} = \frac{i}{2}\left(e^{-ix}-e^{ix}\right)$

$$e^{ix} = \cos x + i\sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

Now, write
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\left(\frac{e^{\frac{in\pi x}{L}}+e^{-\frac{in\pi x}{L}}}{2}\right) + i\,b_n\left(\frac{e^{-\frac{in\pi x}{L}}-e^{\frac{in\pi x}{L}}}{2}\right)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2}e^{\frac{in\pi x}{L}} + \frac{a_n+ib_n}{2}e^{-\frac{in\pi x}{L}}$$
$$= C_0 + \sum_{n=1}^{\infty} C_n e^{\frac{in\pi x}{L}} + C_{-n}e^{-\frac{in\pi x}{L}}$$

Therefore, $\boxed{C_n = \dfrac{a_n - ib_n}{2}, \quad C_{-n} = \dfrac{a_n+ib_n}{2}}$ (This works for $n=0$ too!)

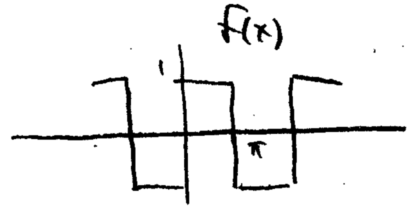
Let's go the other direction:

$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n e^{\frac{in\pi x}{L}} + C_{-n}e^{-\frac{in\pi x}{L}}$$
$$= C_0 + \sum_{n=1}^{\infty} C_n\left(\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L}\right) + C_{-n}\left(\cos\frac{n\pi x}{L} - i\sin\frac{n\pi x}{L}\right).$$
$$= C_0 + \sum_{n=1}^{\infty} (C_n + C_{-n})\cos\frac{n\pi x}{L} + (C_n - C_{-n})\,i\sin\frac{n\pi x}{L}.$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}$$

Therefore, $\boxed{a_n = C_n + C_{-n}, \quad b_n = i(C_n - C_{-n})}$ (This works for $n=0$ too!)

Examples:

(5) Compute the complex Fourier series of

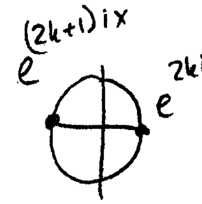


$c_0 = 0$ (average value of $f(x)$).

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} -e^{-inx}\,dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{in} e^{-inx}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{-1}{in} e^{-inx}\right]_{0}^{\pi}$$

$\mathbb{C}$:



$$= \frac{1}{2\pi i n}\left(1 - e^{in\pi} - e^{-in\pi} + 1\right)$$

Note: $e^{in\pi} = e^{-in\pi} = (-1)^n = (-1)^{-n}$.

$$= \boxed{\frac{1}{\pi i n}\left(1-(-1)^n\right)} = \begin{cases} \dfrac{2}{\pi i n} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

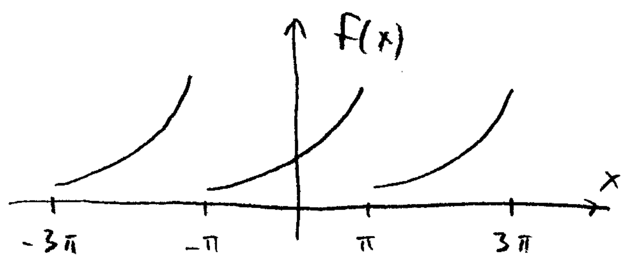
Thus, $f(x) = \sum_{n\in\mathbb{Z}\setminus 0} \frac{1-(-1)^n}{\pi i n} e^{inx} = \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{\pi i n}\left(e^{inx} + e^{-inx}\right)$.

The real coefficients are: $a_n = c_n + c_{-n} = \frac{1-(-1)^n}{\pi i n} + \frac{1-(-1)^n}{-\pi i n} = 0$

$$b_n = i\left(c_n - c_{-n}\right) = i\left(\frac{1-(-1)^n}{\pi i n} - \frac{1-(-1)^n}{-\pi i n}\right) = \frac{2(1-(-1)^n)}{\pi n}.$$

Recall that this is what we computed in Example 1, p. 5.

(6) Compute the complex Fourier series of the 2π-periodic extension of e^x (defined on $[-\pi,\pi]$.)



$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{2\pi} e^x \Big|_{-\pi}^{\pi}$$

$$= \boxed{\frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right)}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx = \frac{1}{2\pi(1-in)} e^{(1-in)x}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(1-in)}\left[e^{(1-in)\pi} - e^{-(1-in)\pi}\right] = \frac{e^{inx}}{2\pi(1-in)}\left[e^{\pi}-e^{-\pi}\right] = \frac{(-1)^n(e^{\pi}-e^{-\pi})}{2\pi(1-in)}$$

<u>Note</u>: $\frac{1}{1-in} = \frac{1}{1-in}\,\frac{1+in}{1+in} = \frac{1+in}{1+n^2} \Rightarrow \boxed{c_n = \frac{(-1)^n(e^{\pi}-e^{-\pi})}{2\pi(1+n^2)}(1+in)}$

The real coefficients are:

$$a_n = c_n + c_{-n} = \frac{(-1)^n(e^{\pi}-e^{-\pi})}{\pi(1+n^2)} \qquad b_n = i(c_n - c_{-n}) = \frac{(-1)^{n+1}n(e^{\pi}-e^{-\pi})}{\pi(1+n^2)}$$

<u>Time/space vs. frequency domains.</u>

Consider a function $f : [-L, L] \longrightarrow \mathbb{C}$. (or equivalently, a $2L$-periodic function.)

We can define f two ways:

* By defining its <u>value</u> $f(x)$ at every $x \in [-L, L]$.
  (This is the <u>spacial</u>, or <u>temporal</u> domain.)
* By defining its Fourier coefficients $\{c_n : n \in \mathbb{Z}\}$.
  (This is the <u>frequency</u> domain.)

<u>Remarkable fact</u>: It is often <u>much</u> easier to define f in terms of frequency, because it's a <u>discrete</u> set of values, rather than in terms of a <u>continuum</u> of values $f(x)$, $x \in [-L, L]$.

Consider a vector $v \in \mathbb{R}^n$, say $v = (a_1, \dots, a_n)$.

$$v \cdot v = (a_1, \dots, a_n) \cdot (a_1, \dots, a_n) = a_1^2 + a_2^2 + \dots + a_n^2 = \|v\|^2$$

There is an analog of this for Fourier series.

<u>Parseval's identity</u>: If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$, then

$$\langle f(x), f(x) \rangle := \frac{1}{L} \int_{-L}^{L} (f(x))^2 \, dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

<u>Proof</u>: $$\frac{1}{L} \int_{-L}^{L} (f(x))^2 \, dx = \frac{1}{L} \int_{-L}^{L} f(x) \underbrace{\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)}_{f(x)} dx$$

$$= \frac{a_0}{2L} \int_{-L}^{L} f(x) \, dx + \frac{1}{L} \int_{-L}^{L} f(x) \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) dx$$

$$= \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} \left(a_n \cdot \underbrace{\frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx}_{a_n} + b_n \cdot \underbrace{\frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx}_{b_n} \right)$$

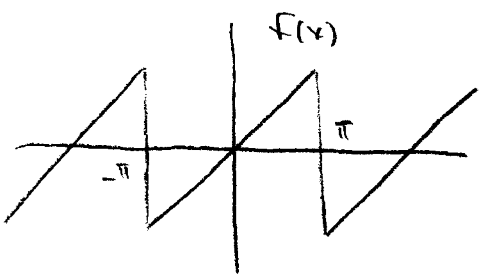
$$= \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

<u>Remark</u>: There is an analog for complex Fourier series:

$$\langle f(x), f(x) \rangle := \frac{1}{2L} \int_{-L}^{L} f(x) \cdot \overline{f(x)} \, dx = \sum_{n=-\infty}^{\infty} c_n^2.$$

<u>Next application</u>: Compute $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$

Let $f(x) = x$ on $[-\pi, \pi]$



In Example 2, p. 6-7, we computed

$a_n = 0$ (since $f(x)$ is odd)

$b_n = \frac{2}{n}(-1)^{n+1} \Rightarrow b_n^2 = \frac{4}{n^2}$

2d

Apply Parseval's identity: $\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2dx = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{2}{3}\pi^2$ (LHS)

RHS: $\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} \frac{4}{n^2}$.

Equate LHS & RHS: $\sum_{n=1}^{\infty}\frac{4}{n^2} = \frac{2}{3}\pi^2 \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$