5. Boundary value problems & Sturm-Liouville theory

Consider a linear, homogeneous equation:

$$y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \dots + a_2(t)\, y'' + a_1(t)\, y' + a_0(t)\, y = 0$$

As we've seen, the general solution is an n-dimensional vector space.

Such an equation with a set of n initial conditions

$$y^{(n-1)}(0) = C_n, \dots, \; y''(0) = C_2, \; y'(0) = C_1, \; y(0) = C_0$$

is called an initial value problem.

The theory (existence & uniqueness) to initial value problems is well-understood: An n^{th} order IVP with n initial conditions will have one unique solution. This is all linear algebra.

We'll consider a related type of problem, called a boundary value problem (BVP). These are less-understood.

Now, suppose $y(x)$ is a function of position.

Examples:

(1) Solve $y'' = -y$, $y(0) = 0$, $y(\pi) = 0$.

Gen'l sol'n: $y(x) = C_1 \cos x + C_2 \sin x$.

$y(0) = C_1 = 0 \Rightarrow y(x) = C_2 \sin x$.

$y(\pi) = 0$ holds for any C_2. So $\boxed{y(x) = C_2 \sin x}$ (Infinitely many sol'ns)

(2) Solve $y'' = -y$, $y(0) = 0$, $y(\frac{\pi}{2}) = 0$.

Again: $y(x) = C_2 \sin x$, but $y(\frac{\pi}{2}) = C_2 = 0 \Rightarrow \boxed{y(x) = 0}$ (One sol'n.)

(3) Solve $y'' = -y$, $y(0) = 0$, $y(\pi) = 1$.

Again: $y(x) = C_2 \sin x$ but $y(\frac{\pi}{2}) = C_2 \sin \frac{\pi}{2} = 0 = 1$. $\boxed{\text{No sol'ns!}}$

Moral: BVP's can have many, one, or no solutions.

We'll study a certain type of BVP that arises in the study of PDE's:

Example (motivation):

Find all solutions to $\boxed{-y'' = \lambda y, \quad y(0) = 0, \quad y(\pi) = 0}$ $(*)$

Remark: Consider the linear differential operator $L = \frac{d^2}{dt^2}$ (or $L = \partial_{tt}$).

A solution to $y'' = \lambda y$ is some function satisfying $\boxed{Ly = \lambda y}$.

We call such a function an eigenfunction corresponding to the eigenvalue λ.

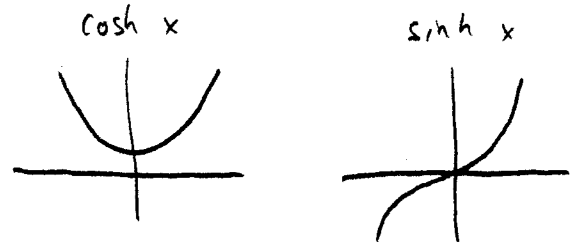
This is motivated by eigenvalues/eigenvectors of matrices: $A\vec{v} = \lambda \vec{v}$.

We want to find all nontrivial solutions to $(*)$.

Case 1: $\boxed{\lambda = 0}$ $y'' = 0 \Rightarrow y(x) = C_1 x + C_2$

$y(0) = 0$ and $y(\pi) = 0 \Rightarrow y(x) = 0$. (no nontrivial sol'ns)

Case 2: $\boxed{\lambda = -\omega^2 < 0}$ $y'' = \omega^2 y \Rightarrow y(x) = A \cosh \omega x + B \sinh \omega x$



$y(0) = A = 0 \Rightarrow y(x) = B \sinh wx$

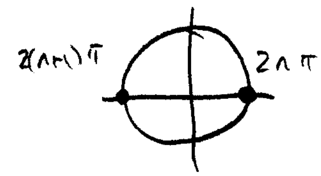
$y(\pi) = B \sinh w\pi = 0 \Rightarrow B = 0$

$\Rightarrow y(x) = 0$ (no non-trivial solutions)

<u>Case 3</u>: $\boxed{\lambda = w^2 > 0}$ $y'' = -w^2 y \Rightarrow y(x) = a \cos wx + b \sin wx$

$y(0) = a = 0 \Rightarrow y(x) = b \sin wx$

$y(\pi) = b \sin w\pi = 0 \Rightarrow w\pi = n\pi \Rightarrow w = n.$



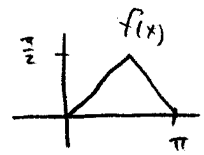
$\sin\theta = 0$ iff $\theta = n\pi$

Thus, $y(x) = b \sin nx$ for any $n \in \mathbb{Z}$. [Recall: $\lambda = -w^2 = -n^2$]

<u>Summary</u>: The differential operator $\frac{d^2}{dx^2}$ in this BVP has:

- <u>eigenvalues</u>: $\lambda_n = -n^2$, for $n = 0, 1, 2, 3, \ldots$
- <u>eigenfunctions</u>: $y_n(x) = \sin nx$

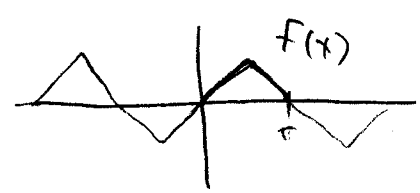
<u>Important observation</u>: Consider the triangle wave on $[0, \pi]$:



We can write $f(x)$ as a sine series: $f(x) = \sum_{n=1}^{\infty} \left[\frac{4}{\pi n^2} \sin\left(\frac{n\pi}{2}\right)\right] \sin nx$,

i.e., as $f(x) = \sum_{n=1}^{\infty} b_n y_n(x)$, where $y_n(x)$ are the <u>eigenfunctions</u> of the BVP $y'' = \lambda y$, $y(0) = y(\pi) = 0$. Since each $y_n(x)$ is a solution, and linearity (superposition) tells us that $f(x)$ is as well!

In other words, the <u>eigenfunctions</u> to the BVP form an <u>orthonormal basis</u> of the solution space! (wrt $\langle f, g \rangle := \frac{2}{L} \int_0^{\pi} f(x) g(x)\, dx.$)



Not surprisingly, this is a special case of a much more general theory, called "Sturm-Liouville theory." Fourier series can be seen as a special case.

Sturm-Liouville theory

A Sturm-Liouville equation is a 2^{nd} order ODE of the following form:

$$-\frac{d}{dx}\left(p(x)\, y'\right) + q(x)\, y = \lambda\, w(x)\, y, \qquad \text{where } p(x), w(x) > 0.$$

We usually are interested in solutions $y(x)$ on a finite interval $[a,b]$, under some homogeneous BC's:

$$\alpha_1\, y(a) + \alpha_2\, y'(a) = 0 \qquad \alpha_1^2 + \alpha_2^2 > 0$$

$$\beta_1\, y(b) + \beta_2\, y'(b) = 0. \qquad \beta_1^2 + \beta_2^2 > 0.$$

Together, this BVP is called a Sturm-Liouville problem (SL problem).

Remark: Consider the linear differential operator $L = \frac{1}{w(x)}\left(-\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)\right)$.

$$\mathbb{C}^\infty \xrightarrow{L_1 = p(x)\frac{d}{dx}} C^\infty \xrightarrow{L_2 = \frac{-1}{w(x)}\frac{d}{dx} + \frac{q(x)}{w(x)}} C^\infty, \qquad L = L_2 \circ L_1$$

$$y \longmapsto p(x)\, y'(x) \longmapsto \frac{-1}{w(x)}\frac{d}{dx}\left[p(x)\, y'(x)\right] + \frac{q(x)}{w(x)}\, y(x)$$

A Sturm-Liouville equation is just an eigenvalue equation: $\boxed{Ly = \lambda y}$

This linear operator has a special property: it is self-adjoint wrt the inner product $\langle f, g\rangle = \int_a^b f(x)\, g(x)\, w(x)\, dx$, which means that

$\boxed{\langle Lf, g\rangle = \langle f, Lg\rangle}$ holds for all f, g satisfying the BC's.

<u>Remarks</u> (about L being "self-adjoint," i.e., $\langle Lf, g\rangle = \langle f, Lg\rangle$.

(i) This follows from integration by parts (it's messy).

(ii) This property of linear operators is analogous to a matrix (a linear map) being <u>symmetric</u>.

(iii) Every 2^{nd} order linear homogeneous ODE, $y'' + P(x)y' + Q(x)y = 0$, can be written as a Sturm-Liouville equation, also called its <u>self-adjoint form</u>.

* (iv) <u>Eigenvectors</u> of self-adjoint operators corresponding to distinct eigenvalues are <u>orthogonal</u>!

<u>Proof</u> of (iv): Suppose L is self-adjoint, and $\lambda_i \neq \lambda_j$ are eigenvalues with eigenvectors $y_i \neq y_j$. (So $Ly_i = \lambda_i y_i$).

We need to show $\langle y_i, y_j\rangle = 0$.

We know:
$$\left.\begin{aligned} \langle Ly_i, y_j\rangle &= \langle \lambda_i y_i, y_j\rangle = \lambda_i \langle y_i, y_j\rangle \\ = \langle y_i, Ly_j\rangle &= \langle y_i, \lambda_j y_j\rangle = \lambda_j \langle y_i, y_j\rangle \end{aligned}\right\}\text{ Subtract these!}$$

Thus, $\lambda_i\langle y_i, y_j\rangle - \lambda_j \langle y_i, y_j\rangle = 0$

$\Rightarrow (\lambda_i - \lambda_j)\langle y_i, y_j\rangle = 0 \Rightarrow \langle y_i, y_j\rangle = 0$ (because $\lambda_i \neq \lambda_j$). $\square$

<u>Goal</u>: Given a Sturm-Liouville problem, $Ly = \lambda y$ (with BC's)

- Find its eigenvalues
- Find its eigenfunctions (which are orthogonal!).

Example: $-y'' = \lambda y$, $y(0)=0$, $y(\pi)=0$ is a SL problem

Note: $p(x)=1$, $q(x)=0$, $w(x)=1$, $\alpha_1=\beta_1=1$, $\alpha_2=\beta_2=0$.

Eigenvalues: $\lambda_n = n^2$, $n=1,2,3,\dots$

Eigenfunctions: $y_n(x) = \sin nx$

Main Theorem: Given a Sturm-Liouville problem:

(a) The eigenvalues are real, and can be ordered so $\lambda_1 < \lambda_2 < \lambda_3 \cdots \to \infty$

(b) Each eigenvalue λ_i has a unique (up to scalars) eigenfunction $y_i(x)$.

(c) With respect to the inner product $\langle f, g\rangle := \int_a^b f(x)\,g(x)\,w(x)\,dx$, the eigenvectors form an orthogonal basis on the set of continuous* functions on $[a,b]$ that satisfy the boundary conditions!

What this means: Assume the eigenfunctions $\{y_n(x)\}$ are orthonormal (i.e., normalize them if $\|y_i\| \neq 1$ for any i)

Now, any continuous function $f(x)$ on $[a,b]$ satisfying the SL boundary conditions can be written uniquely as

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \qquad c_n = \langle f(x), y_n(x)\rangle = \int_a^b f(x)\,y_n(x)\,w(x)\,dx$$

Remark: If $\|y_n(x)\| \neq 1$, then we can write

$$f(x) = \sum_{n=1}^{\infty} c_n \frac{y_n(x)}{\|y_n(x)\|}, \qquad c_n = \left\langle f(x), \frac{y_n(x)}{\|y_n(x)\|}\right\rangle = \frac{1}{\|y_n(x)\|}\langle f(x), y_n(x)\rangle$$

$$= \frac{\langle f(x), y_n(x)\rangle}{\langle y_n(x), y_n(x)\rangle^{1/2}} = \frac{\int_a^b f(x)\,y_n(x)\,w(x)\,dx}{\left(\int_a^b (y_n(x))^2\,w(x)\,dx\right)^{1/2}}$$

Example: Consider the SL problem $-y'' = \lambda y$, $y(0)=0$, $y(1)+y'(1)=0$

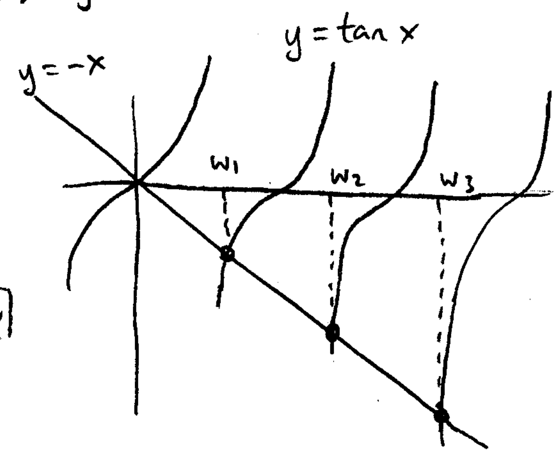
Case 1: $\boxed{\lambda = 0}$: $y''=0$, $y(0)=0$, $y(1)+y'(1)=0$

$\Rightarrow y(x)=0$ (Exercise), i.e., no nontrivial solutions

Case 2: $\boxed{\lambda = -\omega^2 < 0}$ $y''=\omega^2 y$, $y(0)=0$, $y(1)+y'(1)=0$

$\Rightarrow y(x)=0$ (Exercise), i.e., no nontrivial solutions.

Case 3: $\boxed{\lambda = \omega^2 > 0}$ $y''=-\omega^2 y$, $y(0)=0$, $y(1)+y'(1)=0$



General sol'n: $y(x) = a\cos\omega x + b\sin\omega x$

$y(0) = a = 0 \Rightarrow y(x) = b\sin\omega x$

$y(1)+y'(1) = b\sin\omega + b\,\omega\cos\omega = 0$

$\Rightarrow \sin\omega + \omega\cos\omega = 0 \Rightarrow \boxed{\omega = -\tan\omega}$

The eigenvalues are $\boxed{\lambda_n = \omega_n^2}$, where

$\omega_1, \omega_2, \ldots$ are the positive roots of $x = -\tan x$.

[Wolfram Alpha: $\omega_1 = 2.029$, $\omega_2 = 4.913$, $\omega_3 = 7.979, \ldots$]

The eigenfunctions are $\boxed{y_n(x) = \sin\omega_n x}$

Note that this SL equation in "self-adjoint form" has $p(x)=1$, $q(x)=0$, $\boxed{w(x)=1}$

so the eigenfunctions satisfy the following orthogonality relation:

$$\langle y_i(x), y_j(x)\rangle = \int_0^1 y_i(x)\, y_j(x)\, w(x)\,dx = \int_0^1 \sin\omega_i x\, \sin\omega_j x\, dx = 0 \quad \text{if } i \neq j.$$

Moreover, any function $f(x)$, continuous on $[0,1]$, satisfying

$f(0)=0$, $f(1)+f'(1)=0$ can be written uniquely as

(8)

$$f(x) = \sum_{n=1}^{\infty} b_n \frac{\sin w_n x}{\|\sin w_n x\|}, \quad b_n = \left\langle f(x), \frac{\sin w_n x}{\|\sin w_n x\|} \right\rangle = \frac{\int_0^1 f(x) \sin w_n x \, dx}{}$$

$$= \frac{\langle f(x), \sin w_n x \rangle}{\langle \sin w_n x, \sin w_n x \rangle^{1/2}} = \frac{\int_0^1 f(x) \sin w_n x \, dx}{\left(\int_0^1 (\sin w_n x)^2 \, dx \right)^{1/2}}$$

or <u>better</u>: (taking $c_n = \frac{b_n}{\|\sin w_n x\|}$)

$$f(x) = \sum_{n=1}^{\infty} c_n \sin w_n x, \quad c_n = \frac{\langle f(x), \sin w_n x \rangle}{\langle \sin w_n x, \sin w_n x \rangle} = \frac{\int_0^1 f(x) \sin w_n x \, dx}{\int_0^1 (\sin w_n x)^2 \, dx}$$