6. One-dimensional partial differential equations

The study of ordinary differential equations (ODE's) is on the relationship between single-variable functions and their derivatives.

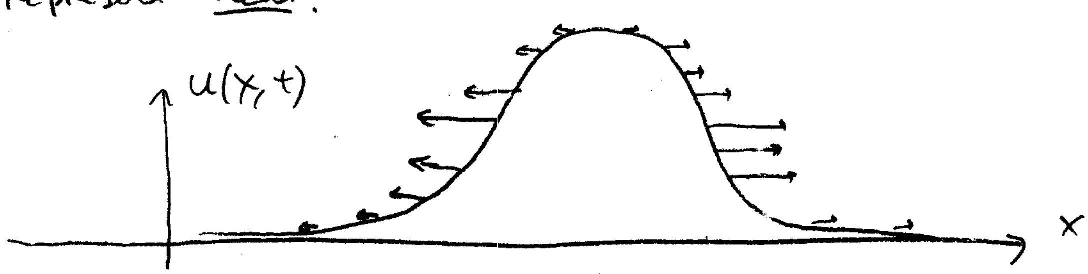
The study of partial differential equations (PDE's) is on the relationship between multi-variate functions and their partial derivatives.

We will begin by studying PDE's of two-variables: position & time.

We'll mainly study the <u>heat equation</u> and the <u>wave equation</u>.

<u>Heat equation</u>:

Consider a material diffusing a 1D domain, and let $u(x,t)$ be the <u>density</u> at position x and time t. For example, $u(x,t)$ might represent <u>heat</u>.



<u>Intuition</u>: Material flows from greater to lesser concentration and the <u>flow</u> at any point in space is <u>proportional</u> to the <u>slope</u> of the curve $u(x,t)$ at that point.

<u>Convention</u>: Denote $\frac{\partial}{\partial t}$ by ∂_t, $\frac{\partial}{\partial x}$ by ∂_x, $\frac{\partial^2}{\partial x^2}$ by ∂_{xx}, etc.

Let $F(x,t)$ be the flow (or flux) at x, t.

$$\text{Then} \quad F(x,t) = -c_1 \partial_x u(x,t), \quad c_1 > 0 \qquad (*)$$

Now, consider the derivative $\partial_x F(x,t)$.

- If $\partial_x F(x,t) > 0$, flow is diverging at this point in space.
  (that is, material is spreading apart, and concentration is decreasing.)
- If $\partial_x F(x,t) < 0$, flow is converging at this point in space.
  (that is, material is crowding closer together; concentration is increasing.)

$$\text{Thus,} \quad \partial_t u(x,t) = -c_2 \partial_x F(x,t), \quad c_2 > 0 \qquad (**)$$

Plugging $(**)$ into $(*)$ yields: $\partial_t u(x,t) = -c_1 c_2 \partial_{xx} u(x,t)$.

This is the heat equation, usually written $\boxed{u_t = c^2 u_{xx}}$

Remark: Heat and temperature are related by $H = \gamma m T$, where γ is the thermal conductivity, and m the mass. Thus, we often say that $u(x,t)$ is the temperature at position x & time t.

Remark: If the "bar" is not uniform, then the heat equation is

$$\rho(x)\sigma(x) u_t = \partial_x \left(\gamma(x) u_x \right), \quad \text{where}$$

$\rho(x) =$ density, $\sigma(x) =$ specific heat, $\gamma(x) =$ thermal conductivity.

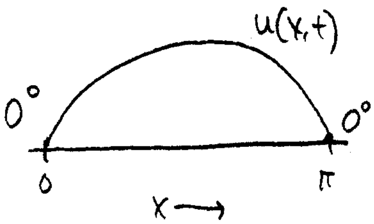
We'll assume these are constant, and so $c^2 = \frac{\gamma}{\rho\sigma}$.

Note that the following functions all solve the heat equation:

$$u(x,t) = (\sin wx)e^{-w^2 t}, \qquad u(x,t) = (\cos wx)e^{-w^2 t}, \qquad u(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}.$$

We'll be interested in solving the heat equation under certain boundary and initial conditions.

<u>Example 1(a)</u>: Let $u(x,t)$ = temp of a bar of length π, insulated along the sides, whose endpoints are kept at $0°$ for all time (<u>boundary</u> conditions), and $u(x,0) = x(\pi - x)$ (<u>initial</u> condition).



Thus, we have the following <u>initial/boundary value problem</u>:

$$u_t = c^2 u_{xx}, \quad u(0,t) = 0, \quad u(\pi,t) = 0, \quad u(x,0) = x(\pi - x).$$

* This is is <u>linear</u> and <u>homogeneous</u>, i.e., if u_1, u_2 are solutions, then so is $c_1u_1 + c_2u_2$. (Physically: "superposition.")

We'll solve this PDE by <u>separation of variables</u>.

Assume there's a solution of the form $\boxed{u(x,t) = f(x)g(t)}$

$\Rightarrow u_t = f(x)g'(t), \quad u_{xx} = f''(x)g(t)$

Boundary conditions: $u(0,t) = f(0)g(t) \Rightarrow f(0) = 0$

$u(\pi,t) = f(\pi)g(t) \Rightarrow f(\pi) = 0$

Plug back in & solve for f & g:

$$u_t = c^2 u_{xx} \implies f(x)\,g'(t) = c^2 f''(x)\,g(t)$$

$$\implies \underbrace{\frac{g'(t)}{c^2 g(t)}}_{\text{doesn't depend on } x} = \underbrace{\frac{f''(x)}{f(x)}}_{\text{doesn't depend on } t} = -\lambda$$ The "eigenvalue equation"

Therefore, this must be <u>constant</u>.

We now have 2 equations:

- A <u>Sturm-Liouville problem</u> for $f(x)$: $f'' = -\lambda f, \quad f(0) = f(\pi) = 0.$
- An <u>ODE</u> for $g(t)$: $g' = c^2 \lambda g$

We know how to solve these: $\lambda_n = n^2, \quad f_n(x) = \sin nx \qquad n = 1, 2, 3, \ldots$

$$g_n(t) = e^{-c^2 n^2 t}$$

Thus for any <u>eigenvalue</u> $\lambda_n = n^2$, we have a solution to the PDE of the form $u_n(x,t) = f_n(x)\,g_n(t) = \sin nx \, e^{-c^2 n^2 t}$.

By <u>linearity</u> (i.e., "superposition"), any linear combination of sol'ns is also a solution. Thus, the general solution is $u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t)$, i.e.,

$$(*) \quad \boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-c^2 n^2 t}}$$

Now, let's solve the <u>initial value problem</u>: $u(x,0) = x(\pi - x)$.

Plug in $t=0$: $u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = x(\pi - x)$ on $[0, \pi]$.

We need to determine the b_n's, so we must write $x(\pi - x)$ as a Fourier sine series.

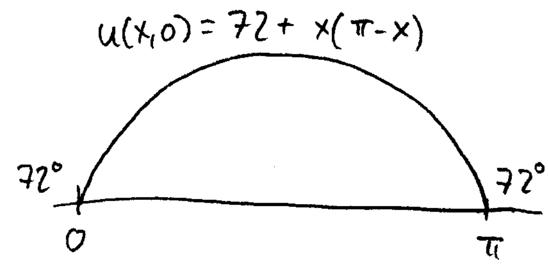
Recall: $b_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin nx\,dx = \frac{4}{\pi n^3}(1-(-1)^n) \Rightarrow b_n = \frac{4(1-(-1)^n)}{\pi n^3}$.

Our particular solution to the initial/boundary value problem is then

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3} \sin nx\, e^{-c^2n^2t}}.$$

Remark: The steady-state solution is $\lim_{t\to\infty} u(x,t) = 0$ (because $e^{-c^2n^2t} \to 0$)

Example 1(b): Consider the same physical situation, but now, say that the boundaries are held fixed at $72°$ (initial condition adjusted accordingly



We now have the following initial/boundary value problem:

$$u_t = c^2 u_{xx}, \quad u(0,t) = u(\pi,t) = 72, \quad u(x,0) = x(\pi - x) + 72.$$

These boundary conditions are inhomogeneous. Make the following substitution:

let $v(x,t) = u(x,t) - 72$. We now have

$v_t = c^2 v_{xx}$, $v(0,t) = v(\pi,t) = 0$, $v(x,0) = x(\pi - x)$, which is the PDE in Example 1(a), the homogeneous equation.

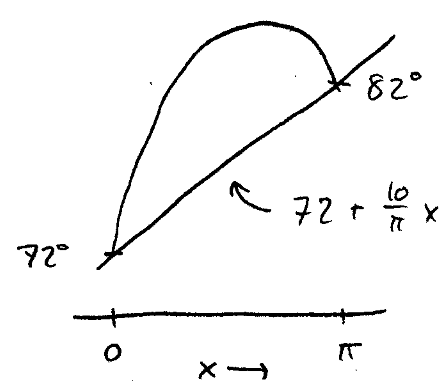
We have $u(x,t) = v(x,t) + 72$

$$\Rightarrow \boxed{u(x,t) = 72 + \sum_{n=1}^{\infty} \frac{4(1+(-1)^n)}{\pi n^3} \sin nx\, e^{-c^2n^2t}}$$

Remark $\lim_{t \to \infty} u(x,t) = 72$ is the <u>steady-state solution</u>.

<u>Big idea</u>: $u(x,t) = u_h(x,t) + u_p(x,t)$, where $u_h(x,t)$ is the solution to the homogeneous equation (including boundary conditions), and $u_p(x,t)$ is <u>any</u> particular solution (e.g., steady-state sol'n.)

<u>Example 1(c)</u>. Consider the same physical situation, but now, say that the left-hand boundary is fixed at 72°, and the right-hand boundary is fixed at 82° (& initial conditions adjusted accordingly.)



We now have the following initial / boundary value problem:

$$u_t = c^2 u_{xx}, \quad u(0,t) = 72, \quad u(\pi, t) = 82, \quad u(x,0) = x(\pi - x) + \left(72 + \tfrac{10}{\pi}x\right)$$

The solution is (not surprisingly)

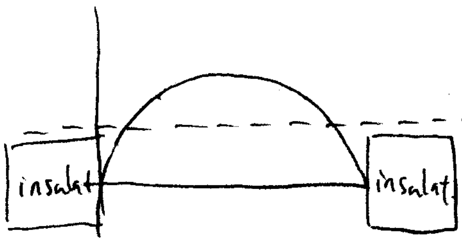
$$\boxed{u(x,t) = \underbrace{\left(72 + \tfrac{10}{\pi}x\right)}_{\text{steady-state sol'n}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin nx \, e^{-c^2 n^2 t}}_{\text{sol'n to the "homogeneous" eq'n.}}}$$

Note that $\lim_{t \to \infty} u(x,t) = 72 + \frac{10}{\pi}x$, the steady-state sol'n.

The boundary conditions $u(0,t) = u(L,t) = 0$ are called <u>Dirichlet</u> boundary conditions. Next, we'll consider <u>Neumann</u> conditions, where $u_x(0,t) = u_x(L,t) = 0$.

Example 2: Same situation as Example 1, but with Neumann BC's:

$$u_t = c^2 u_{xx}, \quad \underbrace{u_x(0,t) = u_x(\pi,t) = 0}_{\text{Represents insulated endpoints through which no heat can pass.}} \qquad u(x,0) = x(\pi - x)$$



steady-state soln = average temp.
(as we'll see, this is $\frac{a_0}{2}$.)

Remark: The only difference is $u_x(0,t) = 0 \Rightarrow f'(0)g(t) = 0 \Rightarrow f'(0) = 0$

$u_x(\pi,t) = 0 \Rightarrow f'(\pi)g(t) = 0 \Rightarrow f'(\pi) = 0$

Assume $u(x,t) = f(x)g(t)$ as before. Plug in and separate variables to get:

- A Sturm-Liouville problem for $f(x)$: $f'' = -\lambda f, \quad f'(0) = f'(\pi) = 0$
- An ODE for $g(t)$: $g' = -c^2 \lambda g$

We know how to solve these: $\lambda_n = n^2, \quad f_n(x) = \cos nx \qquad n = 0, 1, 2, \ldots$

$$g_n(t) = e^{-c^2 n^2 t}$$

Thus, for any eigenvalue $\lambda_n = n^2$, we have a solution to the PDE of the form $u_n(x,t) = f_n(x) g_n(t) = \cos nx \, e^{-c^2 n^2 t}$. (even for $n=0$!)

By linearity, the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} a_n u_n(x,t), \quad \text{i.e.,} \quad \boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, e^{-c^2 n^2 t}}$$

Now, let's solve the initial value problem: $u(x,0) = x(\pi - x)$.

Plug in $t=0$: $u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = x(\pi - x)$ on $[0,\pi]$

We need to determine the b_n's, so we must write $x(\pi-x)$ as a Fourier cosine series.

Recall: $a_0 = \frac{\pi^2}{3}$, $a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi-x)\cos nx\, dx = \frac{2(1-(-1)^n)}{n^2}$.

Our particular solution to the initial/boundary value problem is

$$\boxed{u(x,t) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n^2} \cos nx \, e^{-c^2n^2t}}$$

Remark: The PDE $u_t = c^2 u_{xx}$ assumes perfect insulation along the interior of the bar. If this isn't the case, we get a more complicated eq'n, e.g.,

- $u_t = c^2 u_{xx} + h(x,t)$: External heat source $h(x,t)$
- $u_t = c^2 u_{xx} - ku$: Heat leaks at a rate proportional to the amount of heat at (x,t).

Wave equation: $u_{tt} = c^2 u_{xx}$

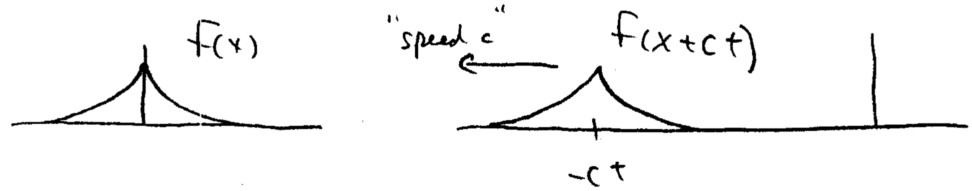
Motivation: Consider the following PDE: $\boxed{u_t = c u_x}$ (*).

Let $f(x)$ be any one-variable function, and set $u(x,t) = f(x+ct)$.

Chain rule $\Rightarrow$ $u_x(x,t) = f'(x+ct)$, $u_t(x,t) = cf'(x+ct)$.

Clearly, u solves (*). Note that $u(x,t) = f(x+ct)$ is a "traveling wave"

Picture of this:



Next, consider the PDE $\boxed{u_t = -c u_x}$ (**)

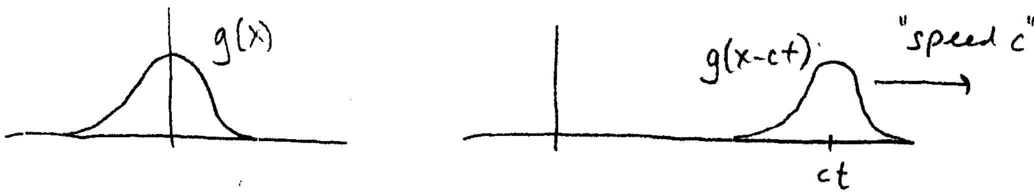
Let $g(x)$ be any one-variable function, and set $u(x,t) = g(x-ct)$.

Chain rule $\Rightarrow$ $u_x(x,t) = g'(x-ct)$, $u_t(x,t) = -c\,g'(x-ct)$.

Clearly, u solves (**).

Note that $u(x,t) = g(x-ct)$ is a "traveling wave" to the right:

Picture of this:



Remark: The PDE (*) can be written as $(\partial_t - c\partial_x)u = 0$

The PDE (**) can be written as $(\partial_t + c\partial_x)u = 0$.

The solutions are those functions in the kernel of the differential operators $\partial_t - c\partial_x$ and $\partial_t + c\partial_x$, respectively.

Now, consider the composition of these operators, i.e., the PDE

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (\partial_{tt} - c^2\partial_{xx})u = 0.$$

This can be written as $\boxed{u_{tt} = c^2 u_{xx}}$ (***).

Check: $u(x,t) = \underbrace{f(x+ct)}_{\substack{\text{killed by}\\ \partial_t - c\partial_x}} + \underbrace{g(x-ct)}_{\substack{\text{killed by}\\ \partial_t + c\partial_x}}$ is a solution.

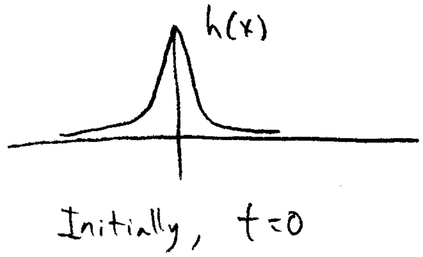
Now, consider the following initial value problem:

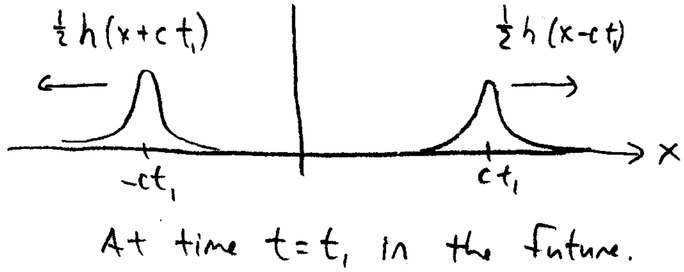
$u_{tt} = c^2 u_{xx}$

$u(x,0) = h(x)$ Initial displacement, or initial wave

$u_t(x,0) = 0$ Initial velocity (vertical, pointwise)

Picture of this: Start with a stationary wave in the ocean, string, etc. Then "let go" at time $t=0$. It should disperse, half the energy to the left, half to the right.



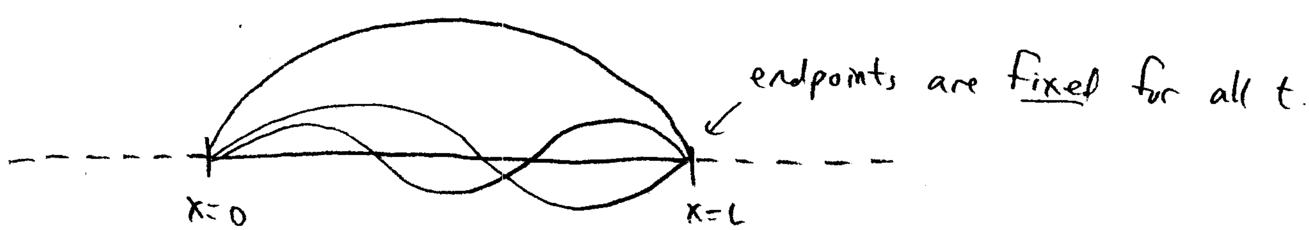


Indeed, the function $u(x,t) = \frac{1}{2} h(x+ct) + \frac{1}{2} h(x-ct)$ solves this IVP. (Check!)

Big idea: $u_{tt} = c^2 u_{xx}$ is the wave equation.

Now, suppose we want to model vibrations (waves) over a finite string/wire of length L.

We need to impose boundary conditions:



Let $u(x,t)$ be the (vertical) displacement at position x & time t.

Fixed endpoints $\Rightarrow$ $u(0,t) = 0$ and $u(L,t) = 0$.

We must specify the initial wave: $u(x,0) = h_1(x)$, <u>and</u> initial (vertical) velocity @ x: $u_t(x,0) = h_2(x)$.

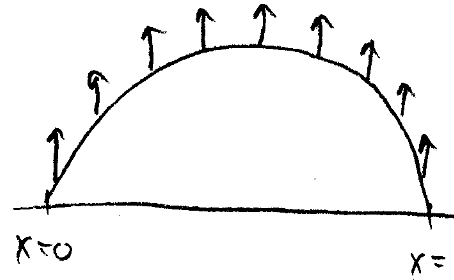
Together, we get an initial/boundary value problem for the <u>wave equation</u>:

$$u_{tt} = c^2 u_{xx}, \quad u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = h_1(x), \quad u_t(x,0) = h_2(x).$$

We can solve this PDE using <u>separation of variables</u>, just like we did for the heat equation. There are only a few slight differences.

<u>Example 3</u>: Consider the PDE

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0, \quad u(\pi,t) = 0 \\ u(x,0) = x(\pi - x), \quad u_t(x,0) = 1. \end{cases}$$



picture at $t=0$

Assume there is a sol'n of the form $u(x,t) = f(x)\,g(t)$; plug back in:

$$u_{tt} = f g'', \quad u_{xx} = f'' g \;\Rightarrow\; f g'' = c^2 f'' g \;\Rightarrow\; \boxed{\frac{f''}{f} = \frac{g''}{c^2 g} = -\lambda}$$

This eigenvalue equation gives us:

- A <u>Sturm-Liouville</u> problem for $f(x)$: $\quad f'' = -\lambda f, \quad f(0) = f(\pi) = 0$
- An <u>ODE</u> for $g(t)$: $\quad g'' = -c^2\lambda^2 g$

We know how to solve these: $\quad \lambda_n = n^2 \quad f_n(x) = \sin nx$

$$g_n(t) = a_n \cos(cnt) + b_n \sin(cnt)$$

Thus, for any eigenvalue $\lambda_n = n^2$, we have a solution to the PDE of the form $u_n(x,t) = f_n(x)\, g_n(x) = \sin nx \left(a_n \cos(cnt) + b_n \sin(cnt) \right)$.

By linearity (i.e, superposition), the general solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(cnt) + b_n \sin(cnt)\right) \sin nx}$$

Now, we need to use both initial conditions:

(i) $u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = x(\pi - x) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3} \sin nx$

$\Rightarrow a_n = \frac{4(1-(-1)^n)}{\pi n^3}$ (The Fourier sine series of $x(\pi - x)$)

(ii) $u_t(x,0) = 1$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-cna_n \sin(cnt) + cnb_n \cos(cnt)\right) \sin nx$$

$$u_t(x,0) = \sum_{n=1}^{\infty} cnb_n \sin nx = 1 = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} \sin nx$$

(The Fourier sine series of 1)

$\Rightarrow cnb_n = \frac{2(1-(-1)^n)}{n\pi} \Rightarrow b_n = \frac{2(1-(-1)^n)}{cn^2\pi}$

The particular solution to the initial/boundary value problem for the wave equation is thus

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[\frac{4(1-(-1)^n)}{\pi n^3} \cos(cnt) + \frac{2(1-(-1)^n)}{\pi c^2 n^2} \sin(cnt) \right] \sin nx}$$

Remark: The PDE $u_t = c^2 u_{xx}$ assumes ideal conditions; no loss of energy or damping, no external force. We can relax these assumptions, e.g.,

- $u_{tt} = c^2 u_{xx} + S(x,t)$: External source; driving force
- $u_{tt} + \gamma_1 u_t + \gamma_0 u = c^2 u_{xx}$: Subject to damping, friction, leakage, etc.

There are many other 1-dimensional PDE's that arise in modeling in the sciences:

- Transport equation (physics) $u_t + c u_x = 0$
- Schrödinger's equation (physics) $i u_t + u_{xx} = 0$
- Telegraph equation (EE) $u_{xx} = c^2 u_{tt} + \lambda_1 u_t + \lambda_2 u$
- Black-Scholes model (Finance) $u_t + \frac{1}{2}\sigma^2 x u_{xx} + r x u_x - r u = 0$
- Airy's equation (physics) $u_t = -c^2 u_{xxx}$
- Bernoulli's beam equation (physics) $u_t = -c^2 u_{xxxx}$