# 7. Higher dimensional partial differential equations

Recall how we derived the heat equation in one dimension:

$u(x,t)$ = density of a material

$F(x,t)$ = flow (or flux) of the material

Intuition: • $F(x,t) = -c_1 \partial_x u(x,t)$, • $\partial_t u(x,t) = -c_2 \partial_x F(x,t)$

Yields $\partial_t u(x,t) = c_1 c_2 \partial_{xx} u(x,t)$, or just $\boxed{u_t = c^2 u_{xx}}$. $(c^2 = c_1 c_2)$

Now, in higher dimensions:

$u(x_1, \dots, x_n, t) = u(\vec{x}, t)$ = density of a material

$\vec{F}(\vec{x}, t)$ = flow (or flux) of the material.

Intuition: • $\vec{F} = -c_1 \nabla u$ ("Fourier's law of heat flow")

• $\partial_t u = -c_2 \operatorname{div} \vec{F}$ (Recall that $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$, and $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$.)

Together, we get $\partial_t u = c^2 \operatorname{div}(\nabla u) = c^2 \nabla \cdot \nabla u$ $(c^2 = c_1 c_2)$.

We write this as $\boxed{u_t = c^2 \nabla^2 u}$. This is the heat equation.

Here, $\nabla^2 := \nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$ is the Laplacian operator, sometimes denoted $\Delta$.

There is a similar derivation for the wave equation. In summary:

✱ In n spatial dimensions, our familiar PDE's are:

- Heat equation: $u_t = c^2 \nabla^2 u$.
- Wave equation: $u_{tt} = c^2 \nabla^2 u$.

Intuitively, <u>steady-state solutions</u> occur for the heat equation, but not for the wave equation. (Heat diffuses, waves propogate.)

<u>Remark:</u> "Steady-state" means that $u_t = 0$. Solutions to the heat equation approach this steady-state solution because "eventually, the temperature doesn't change w.r.t. time.

<u>Key idea:</u> All steady-state solutions satisfy $0 = u_t = c^2 \nabla^2 u$,

i.e., $\boxed{\nabla^2 u = 0} \Rightarrow \dfrac{\partial^2 u}{\partial x_1^2} + \cdots + \dfrac{\partial^2 u}{\partial x_n^2} = 0.$

This PDE is called <u>Laplace's equation</u>.

<u>Def:</u> A function $u$ is called <u>harmonic</u> if $\nabla^2 u = 0$.

<u>Example:</u> $f(x,y) = x^2 - y^2$ is harmonic.

$$f_{xx} = 2, \quad f_{yy} = -2 \quad \Rightarrow \quad \nabla^2 f = f_{xx} + f_{yy} = 2 - 2 = 0. \quad \checkmark$$

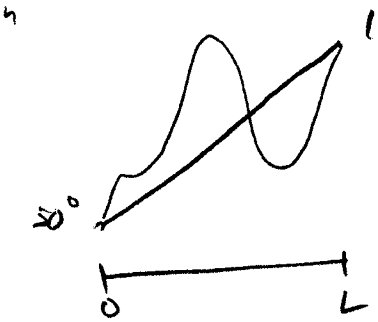
<u>Visualizing harmonic functions:</u>

* <u>Big idea:</u> Harmonic functions are "as flat as possible".

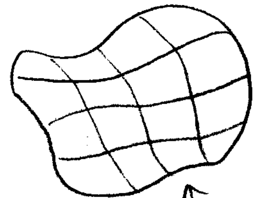
$$\boxed{\begin{array}{c}\text{Harmonic functions;}\\ \nabla^2 u = 0\end{array}} \xleftrightarrow[\text{(set } u_t = 0)]{\text{same}} \boxed{\begin{array}{c}\text{Steady-state sol'ns}\\ \text{to the heat eq'n,}\\ u_t = c^2 u_{xx}\end{array}} \xleftrightarrow[\text{(physical intuition)}]{\text{same}} \boxed{\begin{array}{c}\text{"Flat" functions, i.e.,}\\ \text{"soap bubble surfaces"}\end{array}}$$

same, by transitivity.

In 1D: Consider the temperature $u(x,t)$ of a bar with $u(0,t) = 50$, $u(L,t) = 100$. The steady-state solution satisfies $0 = u_t = c^2 u_{xx}$, so it is a line $u_{ss}(x) = ax+b$, regardless of initial condition.




Physical interpretation: Consider a wire bent in a circular shape, and then dip it into a bucket of soap (or stretch plastic wrap over it)

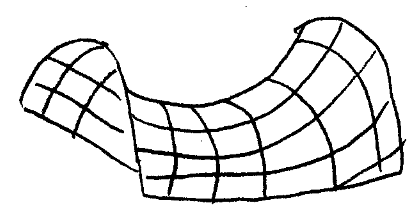



* The surface is a harmonic function! (as flat as possible.)

Fact: If $f$ is harmonic, then for any closed bounded region $R$, $f$ achieves its min & max values on the boundary, $\partial R$.

(Picture "cutting" the graph with a cookie cutter.)

Example: $f(x) = x^2 - y^2$. Graph:



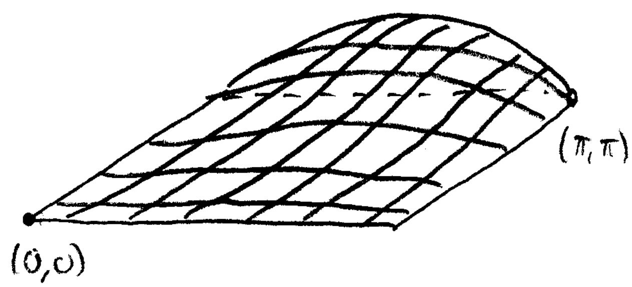
Goal: Solve $u_t = c^2 \nabla u$ subject to boundary & initial conditions.

We'll only do this for a 2D region, so $u_t = c^2(u_{xx} + u_{yy})$.

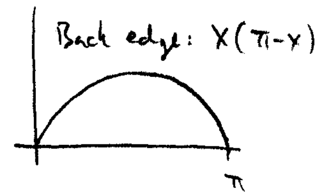
As before, $u(x,y,t) = u_h(x,y,t) + u_{ss}(x,y)$, so our first step will be to solve Laplace's equation: $u_{xx} + u_{yy} = 0$ (to get $u_{ss}$.)

(4)

Example 1(a): Let $u(x,y)$ be a 2-variable function defined for $0 \le x, y \le \pi$ that satisfies the following BVP:




$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = u(\pi,y) = u(x,0) = 0 \\ u(x,\pi) = x(\pi - x) \end{cases}$$




* Physical situation: $u(x,y)$ is the steady-state solution of the 2D heat equation, where 3 sides are fixed at $0°$, and one at $u(x,\pi) = x(\pi - x)$.

We'll solve this by separation of variables:

Assume $u(x,y) = X(x)Y(y)$. $\quad u_{xx} = X''Y, \quad u_{yy} = XY''$.

Our "zero-boundary conditions" imply: $X(0) = X(\pi) = Y(0) = 0$.

Plug back into the PDE:

$$u_{xx} + u_{yy} = X''Y + XY'' = 0 \implies \boxed{\underbrace{\frac{X''}{X}}_{\text{independent of } y} = \underbrace{\frac{-Y''}{Y}}_{\text{independent of } x} = -\lambda}$$

We have: (i) A Sturm-Liouville problem for $X$: $\quad X'' = -\lambda X, \quad X(0) = X(\pi) = 0$

(ii) An ODE for $Y$: $\quad Y'' = \lambda X, \quad Y(0) = 0$.

We can solve these: $\lambda = n^2, \quad \boxed{X_n(x) = \sin nx} \quad n = 1, 2, 3, \dots$

$Y_n(y) = a_n \cosh ny + b_n \sinh ny, \quad Y_n(0) = a_n = 0 \implies \boxed{Y_n(y) = \sinh ny}$

The general solution is $u(x,y) = \sum_{n=1}^{\infty} X_n(x)\, Y_n(y) = \sum_{n=1}^{\infty} b_n \sin nx \sinh ny$.

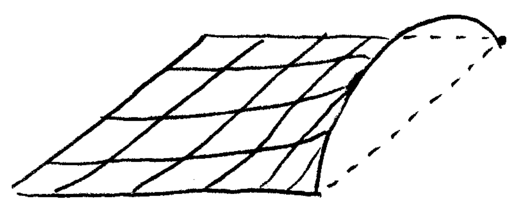
Finally, use the $4^{th}$ boundary condition (plug in $y=\pi$).

$$u(x,\pi) = \sum_{n=1}^{\infty} \left(b_n \sinh n\pi\right) \sin nx = x(\pi - x) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3} \sin nx$$

Equate coefficients: $b_n \sinh n\pi = \dfrac{4(1-(-1)^n)}{\pi n^3} \Rightarrow b_n = \dfrac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi}$.

Our final solution is thus $\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi} \sin nx \sinh ny}$

Example 1(b): Consider the following BVP:

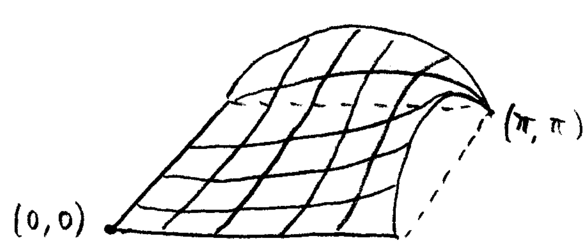


$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = u(x,\pi) = u(0,y) = 0 \\ u(\pi,y) = y(\pi - y) \end{cases}$$

This is exactly the same problem as Example 1(a), but the roles of $x$ & $y$ are reversed.

Thus, by symmetry, the solution is: $\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi} \sinh nx \sin ny}$

Example 1(c): The following BVP is a "superposition" of Examples 1(a) & (b).




$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = u(0,y) = 0 \\ u(x,\pi) = x(\pi - x), \quad u(\pi,y) = y(\pi - y). \end{cases}$$

The solution to this BVP is the sum of the sol'ns to Examples 1(a) & (b).

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh \pi n}\Big[(\sin nx + \sinh ny) + (\sinh nx + \sin ny)\Big]}$$

Physically, this is just the principle of superposition. Mathematically, it's just linearity.

<u>Heat equation in 2D</u>: $u_t = c^2(u_{xx} + u_{yy})$.

To solve it (with initial & boundary conditions):

(i) Find the <u>steady-state solution</u> by solving Laplace's eqn, $\nabla^2 u = 0$, with the same boundary conditions

(ii) Solve the related <u>homogeneous equation</u>. (The PDE with the steady-state solution subtracted off. The boundary conditions will be zero.)

(iii) Add these together: $u(x,y,t) = u_h(x,y,t) + u_{ss}(x,y)$.

<u>Example 2(a)</u>: Let $u(x,y,t)$ = temp. of a square region, $0 \le x, y \le \pi$, subject to $u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0$ (<u>Boundary</u> fixed at $0°$).

and $u(x,y,0) = 2\sin x \sin 2y + 3 \sin 4x \sin 5y$ (<u>Initial</u> heat dist.)

Solve by <u>separation of variables</u>: Assume a solution has the form

$$u(x,y,t) = f(x,y)\, g(t)$$

function of <u>position</u> — function of <u>time</u>

Plug back into the heat equation: $u_t = c^2 \nabla^2 u = c^2(u_{xx} + u_{yy})$

Note that $u_t = f g'$, $\nabla^2 u = \nabla^2 f \cdot g = (f_{xx} + f_{yy})\, g$

function of position   function of time.

$$\frac{f g''}{c^2 f g} = \frac{c^2 \nabla^2 f\, g}{c^2 f g} \Rightarrow \frac{g''}{c^2 g} = \frac{\nabla^2 f}{f} = -\lambda$$

We get 2 equations:

(i) A PDE for $f$: $\nabla^2 f = -\lambda$, $\underbrace{f(0,y) = f(\pi, y) = f(x,0) = f(x,\pi) = 0}_{\text{boundary conditions!}}$

(ii) An ODE for $g$: $g'' = -c^2 \lambda g$

* Solve for $f$: $f_{xx} + f_{yy} = -\lambda$   "Helmholz equation."

Separate variables: Assume $f(x,y) = X(x)\,Y(y)$

Plug back in: $X''Y + XY'' = \lambda XY \Rightarrow$

Divide by $XY$: $\frac{X''Y + XY''}{XY} = \frac{\lambda XY}{XY} \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = -\lambda.$

Rewrite as: $\underbrace{\frac{X''}{X}}_{\text{depends only on } x} = \underbrace{-\lambda - \frac{Y''}{Y}}_{\text{depends only on } y} = -\nu$   must be constant.

We get 2 ODEs: 

$X'' = -\nu X \qquad X(0) = X(\pi) = 0$

$Y'' = \underbrace{(\lambda - \nu)}_{\text{call this } -\mu} Y \qquad Y(0) = Y(\pi) = 0$

call this $-\mu$, so $\boxed{\lambda = \nu + \mu}$

We know: $\nu = n^2$, $X_n(x) = \sin nx \quad n = 1, 2, 3, \ldots$

$\nu = m^2$, $Y_m(y) = \sin my \quad m = 1, 2, 3, \ldots$

Thus, for any $n, m \in \mathbb{N}$, we have a sol'n $f_{nm}(x,y) = \sin nx \sin my$ to the Helmholz equation. Also, $\boxed{\lambda = n^2 + m^2}$

* Solve for $g$: $g' = -c^2(n^2+m^2)\,g$.

$$g_{nm}(t) = e^{-c^2(n^2+m^2)t}$$

Thus, for every $n, m \in \mathbb{N}$, we have a solution of the form

$$u_{nm}(x,y,t) = f_{nm}(x,y)\,g_{nm}(t) = \sin nx \sin my \; e^{-c^2(n^2+m^2)t}.$$

The general solution is, by <u>superposition</u> (i.e., linearity),

$$\boxed{u(x,y,t) = \sum_{n,m\geq 1} b_{nm} \sin nx \sin my \; e^{-c^2(n^2+m^2)t}}$$

<u>Remark</u>: We could write $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$ instead of $\sum_{n,m\geq 0}$; they're the same.

Finally, we use the initial condition (plug in $t=0$):

$$u(x,y,0) = \sum_{n,m\geq 1} b_{nm} \sin nx \sin my = \underline{\underline{2}} \sin x \sin 2y + \underline{\underline{3}} \sin 4x \sin 5y$$

Equate coefficients: $b_{12} = 2$, $b_{45} = 3$, all other $b_{nm} = 0$.

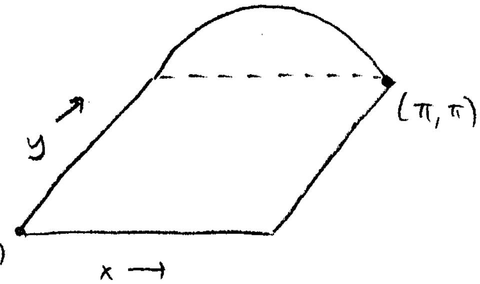
Thus, the particular solution satisfying the BC's & IC's is

$$\boxed{u(x,y,t) = 2 \sin x \sin 2y \; e^{-5c^2t} + 3 \sin 4x \sin 5y \; e^{-41c^2t}}$$

<u>Remark</u>: The <u>steady-state</u> solution is $\lim_{t\to\infty} u(x,y,t) = 0$.

<u>Example 2(b)</u>: Heat equation with <u>non-zero</u> boundary conditions

$$\begin{cases} u_t = c^2 \nabla^2 u \\ u(0,y,t) = u(\pi,y,t) = u(x,0,t) = 0 \\ u(x,\pi,t) = x(\pi - x) \\ u(x,y,0) = 2\sin x \sin 2y + 3\sin 4x \sin 5y + h(x,y) \end{cases}$$




Where $h(x,y)$ is the steady-state solution to this PDE.

<u>Remark</u>: This is the (homogeneous) problem from Example 2(a), but with the boundary conditions from Example 1(a) (Laplace equation).

* Thus, the unique solution to this I/BVP is just the <u>sum</u> of the solutions to Example 1(a) (the <u>steady-state</u> solution) and Example 2(a) (the solution to the PDE with <u>homogeneous</u> BC's.)

i.e.,
$$\boxed{u(x,y,t) = \underbrace{2\sin x \sin 2y\, e^{-5c^2 t} + 3\sin 4x \sin 5y\, e^{-41c^2 t}}_{u_h(x,y,t).\ (\text{sol'n to Ex 2(a)})} + \underbrace{\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3 \sinh n\pi} \sin nx \sin ny}_{u_{ss}(x,y) = h(x,y)\ (\text{sol'n to Ex 1(a)})}}$$

<u>Why this works</u>: Define $v(x,y,t) = u(x,y,t) - h(x,y)$.

Substitute this back into the PDE:

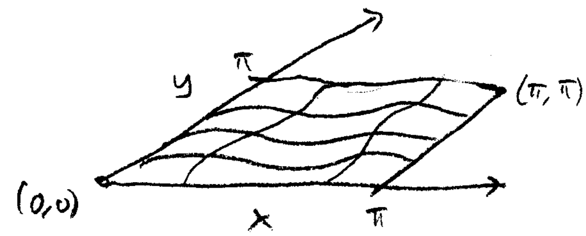
$$\begin{cases} v_t = c^2 \nabla^2 v \\ v(0,y,t) = v(\pi,y,t) = v(x,0,t) = v(x,\pi,t) = 0 \\ v(x,y,0) = 2\sin x \sin 2y + 3\sin 4x \sin 5y \end{cases}$$

This is <u>exactly</u> the PDE in Example 2(a)!

Wave equation in 2D: $u_{tt} = c^2(u_{xx} + u_{yy})$.

Now, $u(x,y,t)$ represents the displacement of a point on a square membrane at position $(x,y)$ & time $t$.

Example 3: Consider the following PDE:




$$\begin{cases} u_{tt} = c^2 \nabla^2 u \\ u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0 & \text{(Boundary is immobile)} \\ u(x,y,0) = x(\pi - x)\, y(\pi - y) & \text{Initial wave:} \\ u_t(x,y,0) = 0 & \text{Initial velocity} \end{cases}$$

Remark: Solving this is almost the same as the 2D heat equation. The only difference in the general solution is $g_{nm}(t)$.

Observe: Assume $u(x,y,t) = f(x,y)\, g(t)$

Plug in: $\dfrac{f g''}{c^2 f g} = \dfrac{c^2 \nabla^2 f g}{c^2 f g} \Rightarrow \dfrac{g''}{c^2 g} = \dfrac{\nabla^2 f}{f} = -\lambda$

Get 2 equations: (i) $\nabla^2 f = -\lambda f \quad f(0,y) = f(\pi,y) = f(x,0) = f(x,\pi) = 0$ (Same!)

(ii) $g'' = -c^2 \lambda g, \quad g'(0) = 0 \leftarrow$ [because $u_t(x,y,0) = 0$]

As before, $\lambda = n^2 + m^2, \quad f_{nm}(x,y) = \sin nx \sin my$

For (ii), we have $g'' = -c^2(n^2+m^2) g, \quad g'(0) = 0$.

$$g_{nm}(t) = a_{nm} \cos\left(c\sqrt{n^2+m^2}\, t\right) + b_{nm} \sin\left(c\sqrt{n^2+m^2}\, t\right), \quad g_{nm}'(0) = 0 \Rightarrow \boxed{b_{nm} = 0}$$

Therefore, the general solution to this PDE is

$$\boxed{u(x,y,t) = \sum_{n,m\geq 1} b_{nm} \sin nx \sin my \cos\left(c\sqrt{n^2+m^2}\,t\right)}$$

Finally, we use the other (non-zero) initial condition:

$$u(x,y,0) = \sum_{n,m\geq 1} b_{nm} \sin nx \sin my = \left(\sum_{n=1}^{\infty} B_n \sin nx\right)\left(\sum_{m=1}^{\infty} \beta_m \sin my\right)$$

$$= x(\pi - x)\, y(\pi - y)$$

$$= \left(\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{\pi n^3} \sin nx\right)\left(\sum_{m=1}^{\infty} \frac{4(1-(-1)^m)}{\pi m^3} \sin my\right)$$

Thus, $b_{nm} = B_n \beta_m = \dfrac{16(1-(-1)^n)(1-(-1)^m)}{\pi^2 n^3 m^3} = \begin{cases} \dfrac{64}{\pi^2 n^3 m^3} & n \text{ \& } m \text{ even} \\ 0 & \text{otherwise.} \end{cases}$

The particular solution to this initial/boundary value problem is:

$$\boxed{u(x,y,t) = \sum_{n,m\geq 1} \frac{16(1-(-1)^n)(1-(-1)^m)}{\pi^2 n^3 m^3} \sin nx \sin my \cos\left(c\sqrt{n^2+m^2}\,t\right).}$$

Remarks:

- $\lim_{t\to\infty} u(x,y,t)$ doesn't exist. This makes sense: heat diffuses, but waves propagate forever.
- Fix $(x,y) = (x_0, y_0)$. The result is a cosine wave – simple harmonic motion. So $u(x,y,t)$ is "a plane's worth of simple harmonic oscillators."