8. PDEs in other coordinate systems

Thus far, we've seen the following PDEs:

- Laplace's eq'n: $\Delta u = 0$
- Heat eq'n: $u_t = c^2 \Delta u$
- Wave eq'n: $u_{tt} = c^2 \Delta u$

In rectangular coordinates, $\Delta = \sum_{i=1}^{n} \partial_{x_i}^2 = \nabla \cdot \nabla$

so in 2D, $\Delta u = u_{xx} + u_{yy}$, and in 3D, $\Delta u = u_{xx} + u_{yy} + u_{zz}$

To solve the heat & wave equations in a 2D square region S (say of side-length π), we needed to find the (Dirichlet) eigenvalues of the square (think: fundamental frequencies), as well as the corresponding eigenfunctions (fundamental nodes).

Mathematically, we had to solve the Helmholtz equation:

$$\Delta f = -\lambda f, \quad \text{with } f(0,y) = f(\pi, y) = f(x,0) = f(x,\pi) = 0.$$

Recall that $\lambda = n^2 + m^2$, $n, m \in \mathbb{N}$, and $f_{nm}(x,y) = \sin nx \sin my$.

Moreover the eigenfunctions are orthogonal, in that

$$\langle f_{nm}, f_{k\ell} \rangle := \iint_S f_{nm}(x,y)\, g_{k\ell}(x,y)\, dA = 0 \quad \text{if} \quad (n,m) \neq (k,\ell).$$

Our next goal is to do this in <u>polar coordinates</u>.

Suppose $f(r,\theta)$ is defined on the disk $D = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta < 2\pi\}$.

Let's solve the Helmholtz equation for f, with Dirichlet BC's:

$$\Delta f = -\lambda f, \qquad f(1,\theta) = 0, \quad f(r,\theta+2\pi) = f(r,\theta).$$

First of all, the Laplacian operator in polar coordinates is

$$\Delta = \frac{1}{r}\partial_r + \partial_r^2 + \frac{1}{r^2}\partial_\theta^2,$$ and so our PDE is

$$(*) \quad f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = -\lambda f, \qquad f(1,\theta) = 0; \quad f(r,\theta+2\pi) = f(r,\theta)$$

The possible values of λ_{nm} are the <u>eigenvalues</u> (fundamental frequencies) of the disk, and the corresponding <u>eigenfunctions</u> f_{nm} are the fundamental modes.

To solve, assume $f(r,\theta) = R(r)\,T(\theta)$.

Note that $f(1,\theta) = R(1)\,T(\theta) = 0 \Rightarrow R(1) = 0$

and $f(r,\theta+2\pi) = R(r)\,T(\theta+2\pi) = R(1)T(\theta) = 0 \Rightarrow T(\theta+2\pi) = T(\theta)$.

Plug back in:
$$\frac{R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT''}{\frac{1}{r^2}RT} = \frac{-RT}{\frac{1}{r^2}RT}$$

$$\Rightarrow \frac{T''}{T} = \frac{-r^2R'' - rR' + r^2\lambda}{R} = -\nu$$

We get 2 ODEs:

(i) $T'' = -\nu T$, $\quad T(\theta+2\pi) = T(\theta)$

(ii) $r^2 R'' + rR' - (r^2\lambda + n^2)R = 0$, $\quad R(1) = 0$

We can solve (i): $\nu = -n^2$, $\quad T_n(\theta) = a_n \cos n\theta + b_n \sin n\theta \qquad n = 0,1,2,\ldots$

Let's focus on (ii):

It is convenient to change variables: let $x = \sqrt{-\lambda}\, r$, and we'll write the ODE in terms of a function of x.

By the chain rule, $\frac{dR}{dr} = \frac{dR}{dx}\cdot\frac{dx}{dr} = \sqrt{-\lambda}\,\frac{dR}{dx}$. Also, note that $r = \frac{1}{\sqrt{-\lambda}}\, x$

This, with $r = \frac{1}{\sqrt{-\lambda}}\, x$ yields: $r^2 R''(r) \longrightarrow x^2 R''(x)$ & $rR'(r) \to xR'(x)$.

Our ODE in (ii) becomes (using y in place of R):

$$\boxed{x^2 y'' + xy' + (x^2 - n^2)y = 0, \qquad y(\sqrt{-\lambda}) = 0}$$

This is <u>Bessel's equation</u>, it has been widely studied.

<u>Remark</u>: Dividing through by x, we can write this ODE as

$$-(xy')' - xy = -n^2 x y, \quad y(\sqrt{-\lambda}) = 0 \qquad \text{for } 0 \le x \le \sqrt{-\lambda}$$

This is a (singular) Sturm-Liouville problem!

The eigenvalues are $-n^2$, and corresponding eigenfunctions are the Bessel Functions (of the first kind): $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{n+2k} k!(n+k)!} x^{n+2k}$

This can be derived using the generalized power series method:

If $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$, then $r = \pm n$ and $a_{k+2} = \frac{-1}{(k+2)(k+2+2n)} a_k$.

Remark: The other solution, $J_{-n}(x)$, is unbounded at $x=0$, so is not relevant to our problem.

By Sturm-Liouville theory, the Bessel functions are orthogonal, via

$$\langle J_n(x), J_m(x) \rangle = \int_0^{\sqrt{-\lambda}} J_n(x) J_m(x) \, x \, dx = 0 \qquad \text{for } n \neq m$$

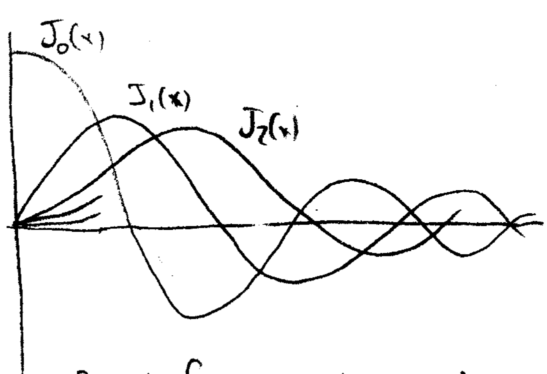
We usually change variables so the domain is $[0,1]$ instead of $[0, \sqrt{-\lambda}]$.

The eigenfunctions become $J_m(\sqrt{-\lambda}\, x)$, and the inner product is redefined as

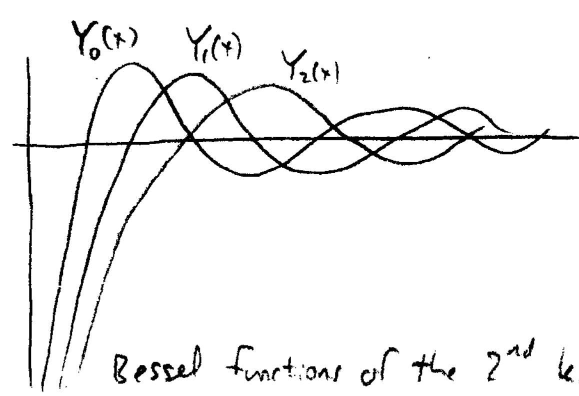
$$\langle J_n(\sqrt{-\lambda}\, x), J_m(\sqrt{-\lambda}\, x) \rangle = \int_0^1 J_n(\sqrt{-\lambda}\, x) J_m(\sqrt{-\lambda}\, x) \, x \, dx = 0 \qquad n \neq m.$$

Remark: The Bessel function satisfy many properties, one in particular being

$$\| J_n(\sqrt{-\lambda}\, x) \|^2 = \int_0^1 J_n^2(\sqrt{-\lambda}\, x) \, x \, dx = \frac{1}{2}\left(J_{n+1}(\sqrt{-\lambda})\right)^2$$



Bessel functions of the 1st kind



Bessel functions of the 2nd kind

Let $J_n(x)$ be the Bessel function of order n, and let ω_{nm} $m=1,2,3,\ldots$

be its positive roots, that is, $J_n(\omega_{nm})=0$.

Recall the Sturm-Liouville form of Bessel's equation:

$$-(xy')'-xy=-n^2xy, \quad y(\sqrt{\lambda})=0.$$

The quantity $\sqrt{-\lambda}=\omega_{nm}$ for some m.

Denote it as $\lambda_{nm}=-\omega_{nm}^2$.

This was the constant from the Helmholtz eqn: $\Delta f=-\lambda f$.

<u>Conclusion</u>: Consider the PDE defined on the unit disk (Dirichlet BC's)

$$\Delta f=-\lambda f, \quad f(1,\theta)=0, \quad f(r,\theta+2\pi)=f(r,\theta).$$

The <u>eigenvalues</u> are $\lambda_{nm}=-\omega_{nm}^2$, where ω_{nm} is the m^{th} positive root of $J_n(r)$, the Bessel function of order n.

The <u>eigenfunctions</u> are $f_{nm}(r,\theta)=\cos n\theta\, J_n(\omega_{nm}r)$, and

$$g_{nm}(r,\theta)=\sin n\theta\, J_n(\omega_{nm}r).$$

These functions form a <u>basis</u> for the solution space of our PDE, so if $h(r,\theta)$ is any solution, we can write

$$h(r,\theta)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}\left[\cos n\theta\, J_n(\omega_{nm}r)\right]+b_{nm}\left[\sin n\theta\, J_n(\omega_{nm}r)\right]$$

$$=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}f_{nm}(r,\theta)+b_{nm}g(r,\theta)$$

By <u>orthogonality</u>, we can derive the coefficients:

$$a_{nm} = \frac{\langle h, f_{nm}\rangle}{\langle f_{nm}, f_{nm}\rangle} = \frac{\iint_D h\cdot f_{nm}\, dA}{\| f_{nm}\|^2} = \frac{2}{\pi J_{n+1}^2(\omega_{nm})}\int_{-\pi}^{\pi}\int_0^1 h(r,\theta)\, J_n(\omega_{nm} r)(\cos n\theta)\, r\, dr\, d\theta$$

$$b_{nm} = \frac{\langle h, g_{nm}\rangle}{\langle g_{nm}, g_{nm}\rangle} = \frac{\iint_D h\cdot g_{nm}\, dA}{\| g_{nm}\|^2} = \frac{2}{\pi J_{n+1}^2(\omega_{nm})}\int_{-\pi}^{\pi}\int_0^1 h(r,\theta)\, J_n(\omega_{nm} r)(\sin n\theta)\, r\, dr\, d\theta.$$

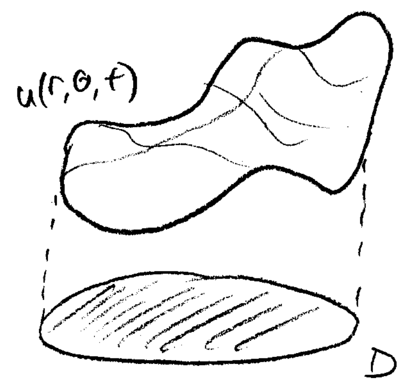
This is called a <u>Fourier-Bessel series</u>. As we've seen, it is natural when dealing with eigenvalues & eigenfunctions of the disk.

We can now solve PDE over circular domains.

<u>Example 1</u> (Heat equation). Let $D = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta < 2\pi\}$.

Consider the following BVP:

$$\begin{cases} u_t = c^2 \Delta u & \\ u(1,\theta,t) = 0, \quad u(r,\theta+2\pi,t) = u(r,\theta,t) & \text{(BC)} \\ u(r,\theta,0) = h(r,\theta) & \text{(IC)} \end{cases}$$



To solve, assume $u(r,\theta,t) = f(r,\theta)\, g(t)$

Plug back in: $\dfrac{fg'}{c^2 fg} = \dfrac{c^2 \Delta f \cdot g}{c^2 fg} \Rightarrow \dfrac{g'}{c^2 g} = \dfrac{\Delta f}{f} = -\lambda$

We get:
- PDE for f: $\Delta f = -\lambda f$, $f(1,\theta) = 0$, $f(r,\theta+2\pi) = f(r,\theta)$
- ODE for g: $g' = -c^2 \lambda g$

We know the solutions to these equations:

- $\lambda_{nm} = w_{nm}^2$, $\quad f_{nm}(r,\theta) = a_{nm}\cos n\theta\, J_n(w_{nm}r) + b_{nm}\sin n\theta\, J_n(w_{nm}r)$.
- $g_{nm}(t) = e^{-c^2 w_{nm}^2 t}$

Thus, the general solution is

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} u_{nm}(r,\theta,t)$$

$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[a_{nm}\cos n\theta\, J_n(w_{nm}r) + b_{nm}\sin n\theta\, J_n(w_{nm}r)\right] e^{-c^2 w_{nm}^2 t}.$$

Finally, we need to use the initial condition:

$$u(r,\theta,0) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}\cos n\theta\, J_n(w_{nm}r) + b_{nm}\sin n\theta\, J_n(w_{nm}r) = h(r,\theta).$$

We can find the coefficients a_{nm} & b_{nm} by orthogonality.

<u>Example 2</u> (Wave equation). Let D be the unit disk, as before. Consider the following BVP:

$$\begin{cases} u_{tt} = c^2\Delta u \\ u(1,\theta,t) = 0, \quad u(r,\theta+2\pi,t) = u(r,\theta,t) & \text{(BC's)} \\ u(r,\theta,0) = h_1(r,\theta) \quad u_t(r,\theta,0) = h_2(r,\theta) & \text{(IC's)} \end{cases}$$

D

The general solution is the same as for the heat equation, except g now satisfies $g'' = -c^2 \lambda g$, $\qquad \lambda = \omega_{nm}^2$

We get $g_{nm}(t) = A_{nm} \cos(c\,\omega_{nm} t) + B_{nm} \sin(c\,\omega_{nm} t)$.

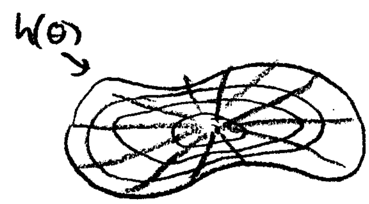
Solving for the coefficients is a bit messier, but the process is the same. We'll skip this.

Now, suppose we want to solve e.g., the heat equation in a disk with non-zero boundary conditions: $u(1,\theta) = h(\theta)$

The solution will be $u(r,\theta,t) = u_h(r,\theta,t) + u_{ss}(r,\theta)$.

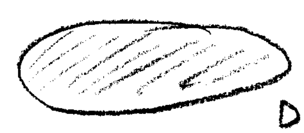
Here, u_{ss} solves <u>Laplace equation</u>: $\Delta u = 0$.

<u>Example 3</u>: Consider the following BVP:



$$\Delta u = 0, \qquad u(1,\theta) = h(\theta), \quad u(r,\theta+2\pi) = u(r,\theta)$$

To solve, assume $u(r,\theta) = R(r)\,T(\theta)$.



Plug back in: (Recall that $\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$)

$$\frac{R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT''}{\frac{1}{r^2}RT} = 0 \Rightarrow -\frac{r^2 R''}{R} - \frac{rR'}{R} = \frac{T''}{T} = -\lambda$$

We get: • $T'' = -\lambda T, \quad T(\theta + 2\pi) = T(\theta)$

(Recall: $\lambda = n^2$, $n = 0,1,2,\ldots$, $T_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$

- $r^2R'' + rR' - n^2R = 0$, $\quad R(1) = 0$, $\quad$ (and $R(0)$ exists).

$\underline{n=0}$: $r^2R'' + rR' = 0 \Rightarrow R(r) = a + b\ln r$.

But $R(0)$ existing $\Rightarrow b = 0 \Rightarrow R(r) = a$.

$\underline{n \neq 0}$: This is a <u>Cauchy-Euler</u> equation. Assume $R(r) = r^k$.

Plug back in: $r^2 k(k-1) r^{k-2} + rkr^{k-1} - n^2 r^k = 0$

$\Rightarrow (k^2 - n^2) r^k = 0 \Rightarrow k = \pm n$.

Thus $R(r) = C_1 r^n + C_2 r^{-n}$.

Again, $R(0)$ existing $\Rightarrow C_2 = 0 \Rightarrow R(r) = C_1 r^n$.

We now have $u_n(r,\theta) = R_n(r) T_n(\theta) = (a_n \cos n\theta + b_n \sin n\theta) r^n$.

Our general solution is $u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n$.

To find the particular solution to this BVP, plug in $r = 1$:

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta = h(\theta).$$

The coeffienets a_n & b_n are the Fourier coefficients of $h(\theta)$.

So far, we've seen PDE's in rectangular & polar coordinates. We can also solve them in other coordinate systems, such as cylindrical (less natural) and spherical (common, e.g., cooling, or electric potential of a sphere).

Spherical coordinates

Let $S = \{(r, \theta, \phi) : 0 \le r \le 1, -\pi \le \theta < \pi, 0 \le \phi \le \pi\}$ be the unit sphere.

The Laplacian operator, in this coordinate system, becomes

$$\Delta f(r,\theta,\phi) = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2 \sin\phi}\partial_\phi^2 + \frac{\cot\phi}{r^2}\partial_\phi + \frac{1}{r^2\sin^2\phi}\partial_\theta^2 .$$

Needless to say, solving the Helmholtz eq'n, $\Delta f = -\lambda f$ to find the eigenvalues & eigenfunctions of the sphere, or solving Laplace's equation, $\Delta f = 0$, is quite treacherous, but can be done.

However, it is quite simple in a few special (natural) cases, such as:

(i) f is zonal: $f_\phi = 0$, i.e., f doesn't depend on longitude.

(ii) f is spherically symmetric: $f_\phi = 0$ and $f_\theta = 0$.

- If f is zonal, then the eigenfunctions involve solutions to Cauchy-Euler ODE's & Legendre's ODE (Legendre polynomials)
- If f is spherically symmetric, $\Delta f = f_{rr} + \frac{2}{r} f_r$, and the eigenfunctions are of the form $f_n(r) = \frac{\sin nr}{r}$ $n = 1, 2, 3 \ldots$