MthSc 434: Advanced Engineering Mathematics (Spring 2012)
Midterm 1
February 21, 2012

NAME: Key

Instructions

- Exam time is 75 minutes
- You may *not* use notes or books.
- Calculators are *not* allowed.
- **Show your work.** Partial credit will be given.

Question	Points Earned	Maximum Points
1		16
2		8
3		8
4		10
5		8
Total		**50**

Student to your left:

Student to your right:

1. Find the general solution to the following differential equations:

   (a) $y' - 2y = 10\cos t$

$y_h(t) = Ce^{2t}$

$y_p(t) = a\cos t + b\sin t$

$y_p'(t) = -a\sin t + b\cos t$

Plug in: $-a\sin t + b\cos t - 2(a\cos t + b\sin t) = 10\cos t$

$(-2a+b)\cos t + (-a-2b)\sin t = 10\cos t + 0\sin t$

$\begin{cases} -2a + b = 10 \\ -a - 2b = 0 \end{cases} \Rightarrow a = -4,\ b = 2$

$\Rightarrow y(t) = y_h(t) + y_p(t) = Ce^{2t} - 4\cos t + 2\sin t$

   (b) $y'' + 9y = 0$

$y(t) = a\cos 3t + b\sin 3t$

(c) $t^2y'' - 2y = 0$

Guess $y(t) = t^r$

$y''(t) = r(r-1)t^{r-2}$

Plug in: $r(r-1)t^r - 2t^r = 0$

$\Rightarrow r^2 - r - 2 = 0 \Rightarrow (r+1)(r-2) = 0 \Rightarrow r = -1, 2$

$y(t) = C_1 e^{-t} + C_2 e^{2t}$

(d) $y'' + 5y' + 6y = 12e^t + 6$

$r^2 + 5r + 6 = (r+2)(r+3) = 0$

$y_h(t) = C_1 e^{-2t} + C_2 e^{-3t}$

$y_p(t) = ae^t + b$

$y_p'(t) = ae^t$

$y_p''(t) = ae^t$

Plug in: $(ae^t + b) + 5ae^t + 6(ae^t + b) = 12e^t + 6$

$12ae^t + 6b = 12e^t + 6 \Rightarrow a = 1,\ b = 1$

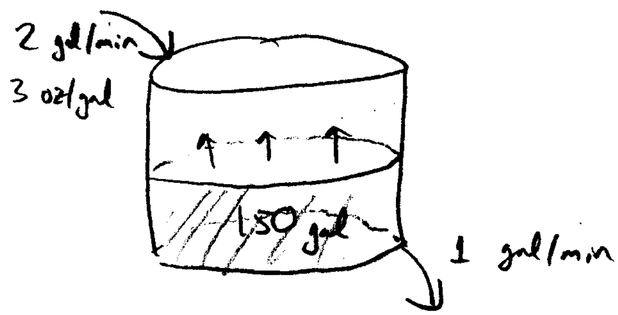
$y(t) = y_h(t) + y_p(t) = C_1 e^{-2t} + C_2 e^{-3t} + e^t + 1$

2. Suppose we have a tank containing 150 gallons of solution with an unknown salt concentration. Suppose water with a salt concentration of 3 oz/gal flows in at a rate of 2 gal/min, and water (fully mixed) drains from the tank at a rate of 1 gal/min. We wish to determine the salt content $x(t)$ at any point in time. Recall that we can do this by writing an ODE:

$$\begin{aligned} x'(t) &= (\text{rate in}) - (\text{rate out}) \\ &= (\text{flow rate in})(\text{incoming concentration}) - (\text{flow rate out})(\text{outgoing concentration}) \\ &= (2 \text{ gal/min})(3 \text{ oz/gal}) - (1 \text{ gal/min})\frac{x(t) \text{ oz}}{150 + t \text{ gal}} \\ &= 6 - \frac{1}{150+t}x(t). \end{aligned}$$

All that is left is solving for $x(t)$.

(a) Sketch this physical situation.



(b) Solve the related homogeneous equation, $x_h' = -\frac{1}{150+t}x_h$.

$$\frac{dx_h}{dt} = -\frac{1}{150+t} x_h$$

$$\int \frac{dx_h}{x_h} = -\int \frac{1}{150+t}\, dt$$

$$\exp\left[\ln x_h = -\ln(150+t) + C \right]$$

$$x_h = e^{-\ln(150+t)} e^{C} = C\left(e^{\ln 150+t}\right)^{-1}$$

$$x_h(t) = \frac{C}{150+t}$$

(c) Using your physical intuition alone, find a simple particular solution, $x_p(t)$. [*Hint*: Consider the situation when there is initially just the right amount of salt so that the salt concentration of the water in the tank never changes.]

$\mathrm{vol}(t) = 150 + t$

If the initial solution has $x_p(0) = 450$ oz of salt, then the concentration will be a constant 3 oz/gal for all time. In this case,

$$x_p(t) = \left(3\,\tfrac{\text{oz}}{\text{gal}}\right)\mathrm{Vol}(t) = 3(150+t) = 450 + t.$$

(d) Find the general solution of this differential equation.

$$x(t) = x_h(t) + x_p(t) = \frac{C}{150+t} + (450+t).$$

3. (a) Carefully define what it means for a set V to be a *vector space* over $\mathbb{R}$.

A set V is a vector space over $\mathbb{R}$ if it is

* closed under addition: $v, w \in V \Rightarrow v+w \in V$.
* closed under scalar multiplication: $v \in V,\ c \in \mathbb{R} \Rightarrow cv \in V$

(b) Carefully define what a means for a differential equation to be *linear*, and *homogeneous*. Give a complete but concise summary about what we've learned regarding the structure of the set of solutions to n^{th} order linear differential equations (both homogeneous and inhomogeneous).

An ODE is linear if it can be written as

$$a_n(t)\, y^{(n)}(t) + \cdots + a_2(t)\, y''(t) + a_1(t)\, y'(t) + a_0(t)\, y(t) = f(t).$$

It is homogeneous if $f(t) = 0$.

* The set of solutions to a linear ODE is an n-dimensional vector space if the ODE is homogeneous, and an n-dimensional affine space if it is inhomogeneous.

4. To solve the differential equation $y'' + t^2y'' - 4ty' + 6y = 0$, assume that the solution has the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$.

   (a) Plug $y(t)$ back into the ODE and determine the recurrence relation for the coefficients of the power series.

$$y(t) = \sum_{n=0}^{\infty} a_n t^n, \quad y'(t) = \sum_{n=0}^{\infty} n a_n t^{n-1}, \quad y''(t) = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$$

Plug in:

$$\underbrace{\sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}}_{\text{shift up 2}} + \sum_{n=0}^{\infty} n(n-1) a_n t^n - \sum_{n=0}^{\infty} 4n a_n t^n + \sum_{n=0}^{\infty} 6 a_n t^n = 0$$

$$\Downarrow$$

$$\overbrace{\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n}$$

Combine terms:

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n^2 - 5n + 6) a_n \right] t^n = 0$$

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+2)(n+1)} a_n$$

(b) **Explictly compute a_n in terms of a_0 and a_1 for $n \leq 5$. How many linear independent *polynomial* solutions are there to this ODE (i.e., up to scalar multiples)? Briefly justify your answer.**

Fix a_0: $a_2 = -\frac{(-2)(-3)}{2\cdot 1} a_0 = -3a_0$ $(n=0)$

$a_4 = 0\, a_2 = 0$ $(n=2)$

Fix a_1: $a_3 = -\frac{(-1)(-2)}{3\cdot 2} a_1 = -\frac{1}{3} a_1$ $(n=1)$

$a_5 = 0\, a_3 = 0$ $(n=3)$

There are two linearly independent polynomial solutions,

because $a_4 = a_6 = a_8 = \cdots = 0$ and $a_5 = a_7 = a_9 = \cdots = 0$.

They are $y_0(t) = a_0 - 3a_0 x^2 = a_0(1-x^2)$

and $y_1(t) = a_1 x - \frac{1}{3} a_1 x^3 = a_1(x - \frac{1}{3}x^3)$

(c) **Write down an explicit basis for the solution space.**

$$\left\{ 1 - x^2,\ x - \tfrac{1}{3}x^3 \right\}$$

(Note: This means that the general solution is

$y(t) = a_0(1-x^2) + a_1(x - \frac{1}{3}x^3)$.)

5. The following differential equation is called *Bessel's equation*:

$$t^2y'' + ty' + (t^2 - \frac{1}{4})y = 0.$$

It is elementary (albeit a bit messy) to show that $\frac{1}{\sqrt{t}}\cos t$ and $\frac{1}{\sqrt{t}}\sin t$ are both solutions. Use this knowledge to find the general solution of this differential equation.

$$y(t) = C_1 \frac{1}{\sqrt{t}} \cos t + C_2 \frac{1}{\sqrt{t}} \sin t$$

Now, consider a more complicated equation:

$$t^2y'' + ty' + (t^2 - \frac{1}{4})y = (t^2 - \frac{1}{4}).$$

Determine if this equation has a steady-state (constant) solution. Also, find the general solution. Explain your reasoning.

Set $y' = 0$: $(t^2 - \frac{1}{4})y = t^2 - \frac{1}{4}$.

Clearly, $y(t) = 1$ is a solution.

General sol'n: $y(t) = y_h(t) + y_p(t) = C_1 \frac{1}{\sqrt{t}} \cos t + C_2 \frac{1}{\sqrt{t}} \sin t + 1$