MthSc 434: Advanced Engineering Mathematics (Spring 2012)
Midterm 2
March 29, 2012

NAME: Key

Instructions

- Exam time is 75 minutes
- You may *not* use notes or books.
- Calculators are *not* allowed.
- **Show your work.** Partial credit will be given.

Question	Points Earned	Maximum Points
1		10
2		10
3		10
4		8
5		12
Total		**50**

Student to your left:

Student to your right:

1. Solve the following differential equations:

   (a) $y' = ty$.

$$\frac{dy}{dt} = ty$$

$$\int \frac{dy}{y} = \int t\, dt$$

$$\ln|y| = \frac{t^2}{2} + C$$

$$|y| = e^{t^2/2 + C}$$

$$\boxed{y(x) = Ce^{t^2/2}}$$

   (b) $y'' + \omega^2 y = 0, \quad y'(0) = y'(\pi) = 0$. (Make sure you determine all possible values of ω, with justification.)

$$y(x) = a\cos wx + b\sin wx$$

$$y'(x) = -aw\sin wx + bw\cos wx$$

$$y'(0) = bw = 0 \Rightarrow b = 0$$

$$y'(\pi) = -aw\sin(w\pi) = 0$$

$$\Rightarrow w\pi = n\pi \Rightarrow \underline{\underline{w = n}}$$ (unless $a = 0$ or $w = 0$ $\Rightarrow y(x) = 0$)

$$\boxed{y(x) = a\cos nx, \quad n = 0, 1, 2, 3, \ldots}$$

2. Consider the sawtooth wave defined as $f(x) = x$ on $[-\pi, \pi]$, and extended to be 2π-periodic. Recall that the Fourier series for f was

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx .$$

Now, solve the differential equation

$$y' + 2y = f(x),$$

where $f(x)$ is the sawtooth wave defined above. [*Hint*: Recall that $y(x) = y_h(x) + y_p(x)$. Take a moment to think about what type of particular solution this equation will have. For example, will it have sine terms, cosine terms, or both?]

$$y(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \qquad y_h(x) = Ce^{-2t}$$

$$y'(x) = \sum_{n=1}^{\infty} -n a_n \sin nx + n b_n \cos nx$$

$$a_0 + \sum_{n=1}^{\infty} (-n a_n \sin nx + n b_n \cos nx) + 2a_n \cos nx + 2b_n \sin nx = f(x)$$

$$a_0 + \sum_{n=1}^{\infty} (2a_n + nb_n)\cos nx + (2b_n - na_n)\sin nx = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

$$a_0 = 0, \quad \begin{cases} nb_n + 2a_n = 0 \\ 2b_n - na_n = \frac{2}{n}(-1)^{n+1} \end{cases}$$

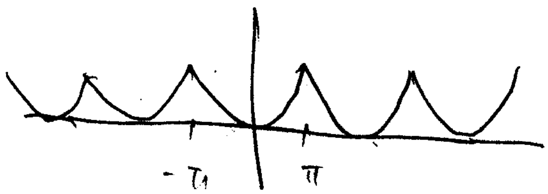
$$\begin{array}{l} 2nb_n + 4a_n = 0 \\ -\left(2nb_n - n^2 a_n = 2(-1)^{n+1}\right) \\ \hline (4+n^2)a_n = 2(-1)^n \end{array} \qquad \begin{array}{l} n^2 b_n + 2n a_n = 0 \\ +\left(4b_n - 2na_n = \frac{4}{n}(-1)^{n+1}\right) \\ \hline (4+n^2)b_n = \frac{4}{n}(-1)^{n+1} \end{array}$$

$$a_n = \frac{2(-1)^n}{4+n^2} \qquad b_n = \frac{4}{n(4+n^2)}(-1)^{n+1}$$

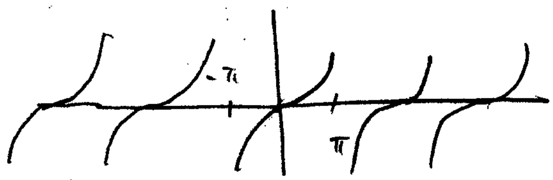
$$\boxed{y(x) = \underbrace{Ce^{-2t}}_{y_h(x)} + \underbrace{\sum_{n=1}^{\infty} \frac{2(-1)^n}{4+n^2} \cos nx + \frac{4(-1)^{n+1}}{n(4+n^2)} \sin nx}_{y_p(x)}}$$

3. Consider the function $f(x) = x^2$, for $0 \le x \le \pi$.

   (a) Sketch the odd and even extensions of $f(x)$ (for at least $-5\pi \le x \le 5\pi$).



even



odd

   (b) Compute the Fourier *sine series* of $f(x)$. You many use the fact that

$$\int x^2 \sin nx \, dx = \frac{2x\sin(nx)}{n^2} + \frac{(2-n^2x^2)\cos(nx)}{n^3}.$$

$$b_n = \frac{2}{\pi}\int_0^\pi x^2 \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\cancel{\frac{2x\sin nx}{n^2}}^{\,0} + \frac{(2-n^2x^2)\cos(nx)}{n^3}\right]_0^\pi$$

$$= \frac{2}{\pi n^3}\left((2-n^2\pi^2)(-1)^n - 2\right)$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n^3}\left((2-n^2\pi^2)(-1)^n - 2\right)\sin nx}$$

7

4. For this problem, you will be graded on *both* the accuracy and the quality of your explanation. Be complete, concise, and clear.

   (a) Carefully define what it means for a set V to be a *vector space* over $\mathbb{R}$, and what it means for a set A to be an *affine space*.

V is a vector space if it is closed under addition and scalar multiplication.

A is an affine space if for some (actually, any), $w \in A$, the set

$\{a-w \mid a \in A\}$ is a vector space.

   (b) Carefully define what a means for a 2nd order differential equation to be *linear*, and when such a linear equation is *homogeneous*. Give a complete but concise summary about what we've learned regarding the structure of the set of solutions to 2nd order linear differential equations (both homogeneous and inhomogeneous).

A 2^{nd} order ODE is linear, if it can be written as

$$y'' + a(t)\,y' + b(t)\,y = f(t),$$

and homogeneous if $f(t) = 0$.

The set of solutions to a 2^{nd} order linear equation form either a vector space (if its homogeneous) or an affine space (if it's inhomogeneous)

5. Solve the differential equation $2xy'' + y' + y = 0$, by looking for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

That is, find r, and find the recurrence relation for the coefficients.

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \qquad y''(x) = \sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2}$$

$$\sum_{n=0}^{\infty} 2(n+r-1)(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$x^r \left[\sum_{n=0}^{\infty} (2n+2r-1)(n+r) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \right] = 0$$

$$\sum_{n=-1}^{\infty} (2n+2r+1)(n+r+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(2r-1) r a_0 + \sum_{n=0}^{\infty} \left[(2n+2r+1)(n+r+1) a_{n+1} + a_n \right] x^n = 0$$

$$\boxed{r = \tfrac{1}{2}, 0} \qquad \boxed{a_{n+1} = \frac{-1}{(2n+2r+1)(n+r+1)} a_n}$$

Useful information

- If we define the following inner product on the space of real-valued $2L$-perioic functions

$$\langle f, g\rangle = \frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx\,,$$

  then the real Fourier series of a $2L$-periodic function $f(x)$ is the orthogonal expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

  where

$$a_n = \langle f(x), \cos\frac{n\pi x}{L}\rangle\,, \qquad b_n = \langle f(x), \sin\frac{n\pi x}{L}\rangle\,.$$

- If we define the following inner product on the space of complex-valued $2L$-perioic functions

$$\langle f, g\rangle = \frac{1}{2L}\int_{-L}^{L} f(x)\overline{g(x)}\,dx\,,$$

  then the complex Fourier series of a $2L$-periodic function f is the orthogonal expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi n x/L}$$

  where $c_n = \langle f(x), e^{i\pi n x/L}\rangle$.

- The real and complex Fourier coefficients are related by

$$a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}), \quad c_n = \frac{a_n + ib_n}{2}, \quad c_{-n} = \frac{a_n - ib_n}{2}.$$

- For any integer n: $\cos n\pi = (-1)^n$, $e^{in\pi} = e^{-in\pi} = (-1)^n$, $\sin n\pi = 0$.

- A function f is even if $f(x) = f(-x)$, and odd if $f(x) = -f(-x)$.

- If $f(x)$ is a function defined on $[0, L]$, then the Fourier cosine series is the Fourier series of the even extension of f, and the Fourier sine series is the Fourier series of the odd extension of f.