Read: Stahl, Chapters 9.1, 9.2.

1. Prove that the rotations $R_{0,\alpha}$ and $R_{0,-\alpha}$ are conjugate in $\mathrm{Isom}(\mathbf{E}^2)$.

2. Prove that the glide reflections γ_{AB} and γ_{CD} are conjugate in $\mathrm{Isom}(\mathbf{E}^2)$ if and only if $d(A,B) = d(C,D)$.

3. Let $f = \gamma_{AB}$, $g = \gamma_{AC}$, where A, B, and C are distinct points of $\mathbf{E}^2$. Describe the Euclidean isometry fgf^{-1} in standard form (i.e., as a translation, rotation, reflection, or glide-relection).

4. Find the image of the points i, $1+i$, and $-3+4i$ when subjected to the following isometries, where $0 = (0,0)$ and $A = (3,0)$.

   (i) $I_{A,2}$ (ii) $f(z) = \dfrac{2z-1}{z+2}$.

5. Express the following compositions as Möbius transformations, where $0 = (0,0)$ and $A = (3,0)$.

   (i) $I_{A,2} \circ I_{0,3}$ (ii) $\dfrac{2(-\bar{z})-1}{(-\bar{z})+2} \circ I_{0,3}$ (iii) $\dfrac{2(-\bar{z})-1}{(-\bar{z})+2} \circ \dfrac{2z-1}{z+2}$.

6. Consider the following subgroups of $\mathrm{GL}_2(\mathbb{R})$, the degree-2 *general linear group*: $\mathrm{GL}_2^+(\mathbb{R})$ consists the matrices with positive determinant, and $\mathrm{SL}_2(\mathbb{R})$, called the *special linear group*, consists of the matrices with determinant 1. The *projective* versions of these groups are obtained by taking the quotient with their respective centers (scalars of the identity matrix). That is,

   $$\mathrm{PGL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})/\langle cI\rangle\,, \qquad \mathrm{PGL}_2^+(\mathbb{R}) = \mathrm{GL}_2^+(\mathbb{R})/\langle cI\rangle\,, \qquad \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}\,.$$

   (a) Prove that the center of the general linear group, $Z(\mathrm{GL}_2(\mathbb{R})) = \langle cI \mid c \in \mathbb{R}^\times\rangle$, is a subset (and hence a a subgroup) of $\mathrm{GL}_2^+(\mathbb{R})$.

   (b) Prove that $\mathrm{PGL}_2^+(\mathbb{R})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$. [*Hint*: Use the first isomorphism theorem – if $\varphi\colon G \to H$ is a homomorphism, then $G/\ker\varphi \cong \mathrm{im}\,\varphi$.]