Read: Stahl, Chapters 9.3, 9.4, 9.5.

1. Let Z be the set of 2×1 nonzero column vectors over $\mathbb{C}$, and let $\mathcal{Z}$ be the set of equivalence classes of Z generated by the equivalence relation that $\mathbf{w} \sim \mathbf{z}$ if $\mathbf{w} = \lambda\mathbf{z}$ for some $\lambda \in \mathbb{C}^{\times}$. Let $\Phi\colon \mathcal{Z} \mapsto \hat{\mathbb{C}}$ be defined by the rule

$$\lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \stackrel{\Phi}{\longmapsto} \begin{cases} \frac{z_1}{z_2} & \text{if } z_2 \neq 0 \\ \infty & \text{otherwise.} \end{cases}$$

   (a) Show that Φ is bijective.

   (b) The group $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathcal{Z}$ by the rule

$$\mathbf{A}.\mathbf{z} = \begin{cases} \mathbf{Az} & \text{if } \det \mathbf{A} > 0 \\ \mathbf{A}\bar{\mathbf{z}} & \text{if } \det \mathbf{A} < 0, \end{cases}$$

   with the convention that $\dfrac{a\infty + b}{c\infty + d} = \dfrac{a}{c}$ and $\dfrac{a}{0} = \infty$. Show that

$$\Phi(\mathbf{A}.\mathbf{z}) = \mu_A(\Phi(\mathbf{z}))\,,$$

   where μ is the Möbius map. [*Hint*: Consider $\mathbf{z} = \begin{pmatrix} z \\ 1 \end{pmatrix}$.]

2. This problem completes the proof of the classification of hyperbolic isometries.

   (a) Show that $\ker \mu = Z(\mathrm{GL}_2(\mathbb{R}))$.

   (b) Prove that $\mathrm{Isom}(\mathbf{H}^2) \cong \mathrm{PGL}_2(\mathbb{R})$.

3. This problem proves that $\mathrm{PGL}_2(\mathbb{R})$ acts 3-transitively on ideal points, and is completely determined by where it sends 0, 1, and ∞.

   (a) Prove that there is an element of $\mathrm{PGL}_2(\mathbb{R})$ which sends $(0, 1, \infty)$ to any three distinct points (a, b, c) of $\hat{\mathbb{R}}$, in that order.

   (b) What can you say about an element of $\mathrm{PGL}_2(\mathbb{R})$ that fixes 0, 1, and ∞?

   (c) Prove that any elements of $\mathrm{PGL}_2(\mathbb{R})$ is uniquely determined by where it sends 0, 1, and ∞.

4. Prove that every ideal triangle has area π.

5. Let f be an orientation-reversing element of $\mathrm{Isom}(\mathbf{H}^2)$ such that $f(\infty) = \infty$.

   (a) Prove that f also fixes a point in $\mathbb{R}$.

   (b) If we additionally know that $f(0) = 0$, prove that $f(z) = a\bar{z}$ for some $a < 0$.

6. Let f be an orientation-reversing element of $\mathrm{Isom}(\mathbf{H}^2)$.

   (a) Prove that f fixes two points in $\hat{\mathbb{R}}$.

   (b) Prove that f is conjugate to some orientation-reversing element of $\mathrm{Isom}(\mathbf{H}^2)$ that fixes both 0 and ∞.