1. Define the triangle $T_{ijk} \subset \mathbb{R}^n$ by

$$T_{ijk} = \{(x_1, \ldots, x_n) \mid x_i + x_j + x_k = 1,\ x_i, x_j, x_k \geq 0,\ x_\ell = 0 \text{ for } \ell \neq i, j, k\}\,.$$

   In this problem, we prove that in $\mathbb{R}^n$, the intersections of the T_{ijk} are precisely given by the set intersections of their indices.

   (a) Prove that $T_{123} \cap T_{456} = \emptyset$.

   (b) Prove that $T_{123} \cap T_{345}$ is a common vertex of the two triangles.

   (c) Prove that $T_{123} \cap T_{234}$ is a common edge of the two trianges.

2. Let S_1 and S_2 be surfaces. Use the cut-and-paste description of the connected sum to prove that $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_3) - 2$.

3. Recall that the fundamental moves on polygon complexes are adding/removing midpoints, adding/removing dividing edges in polygons, and adding/removing hanging edges.

   (a) Prove that an n-gon may be subdivided into triangles using a sequence of fundamental moves.

   (b) Prove that a triangle may be barycentrically subdivided using a sequence of fundamental moves.

4. An *abstract regular polyhedron* is a way of constructing the 2-sphere, S^2, as a polygonal complex such that each polygonal has n sides each vertex is incident to m polygons. Use the combinatorial Gauss-Bonnet theorem to classify the regular polyhedra, with proof.

5. A *trivalent polyhedron* is a polygon complex homeomorphic to S^2 such that exactly three polygons meet at any vertex. Prove that there are only finitely many trivalent polyhedra that can be made from triangles, squares, and pentagons.

6. A triangle is *commensurable* if its three angles are of the form π/n_i, where $n_i \in \mathbb{N}$.

   (a) Find (with proof) the area of the smallest possible commensurable hyperbolic triangle.

   (b) Let S be a surface of genus $g \geq 2$. Prove that S cannot be smoothly triangulated by either spherical or Euclidean triangles.

   (c) Prove that a smooth triangulation of a surface of genus $g \geq 2$ by commensurable triangles contains at most $168(g-1)$ triangles.