3. The hyperbolic plane.

**Def:** The hyperbolic plane \( \mathbb{H}^2 \) consists of

(i) The space \( X = \{ (x, y) \mid y > 0 \} \) (the "upper half-plane.")

(ii) The length function (of a path \( \gamma : [a, b] \to \mathbb{H}^2 \))

\[
|\gamma| = \int_a^b \frac{|\gamma'(t)|}{y(t)} \, dt
\]

where \( \gamma(t) = (x(t), y(t)) \) dt.

(iii) The angle function \( \Delta(\gamma_1(t_1), \gamma_2(t_2)) = \Theta \), where

\[
\cos \Theta = \frac{\gamma_1'(t_1) \cdot \gamma_2'(t_2)}{|\gamma_1'(t_1)| \cdot |\gamma_2'(t_2)|}
\]

(Same as Euclidean angle)

**Note:** This means that \( |\gamma| = \int \frac{\sqrt{dx^2 + dy^2}}{y} \).

Need to check: \( |\gamma| \) is invariant under change in parameter. (Exercise)

**Example:** Consider the points shown here:

\[
|\gamma_{AE}| = \int_{1/10}^{1} \frac{dy}{y} = \ln 1 - \ln \frac{1}{10} \approx 2.303.
\]

\[
|\gamma_{AB}| = \int_{0}^{1} dx = 1
\]

\[
|\gamma_{CD}| = \int_{0}^{1} \frac{dx}{\sqrt{2}} = 2
\]

\[
|\gamma_{EF}| = \int_{0}^{1/10} \frac{dx}{\sqrt{10}} = 10
\]
Prop (Stahl 4.1.1): Let \( g \) be a circle with center \( C = (C, 0) \) and radius \( r \).

If \( P, Q \) are points of \( g \) such that \( CP, CQ \) make angles \( \alpha < \beta \) with the positive \( x \)-axis, then the length of the path \( \gamma_{PQ} \) from \( P \) to \( Q \) along \( g \) is

\[
|\gamma_{PQ}| = \ln\left(\frac{\csc \beta - \cot \beta}{\csc \alpha - \cot \alpha}\right).
\]

Proof: Let \( t \) be the angle to a point \((x, y)\) on \( g \). Then

\[
(x(t), y(t)) = (C + r \cos t, r \sin t)
\]

\[
\Rightarrow \quad dx = -r \sin t \, dt, \quad dy = r \cos t \, dt
\]

\[
|\gamma_{PQ}| = \int_{\alpha}^{\beta} \sqrt{(-r \sin t \, dt)^2 + (r \cos t \, dt)^2} = \int_{\alpha}^{\beta} \frac{r \, dt}{r \sin t}
\]

\[
= \int_{\alpha}^{\beta} \csc t \, dt = \ln\left(\frac{\csc \beta - \cot \beta}{\csc \alpha - \cot \alpha}\right).
\]

Cor: If \( P, Q \) lie on a circle with radius \( r \), center \( C \), and \( P', Q' \) lie on a circle with radius \( r' \) and center \( C' \), and \( C, P, P' \) are collinear, as are \( C, Q, Q' \), then

\[
|\gamma_{PQ}| = |\gamma_{P'Q'}|.
\]
Prop (Stahl 4.1.3): If \(0 < y_1 \leq y_2\), then the path \(\Gamma\) from \((a, y_2)\) to \((a, y_1)\) along a Euclidean vertical line, has length \(|\Gamma| = \ln \left( \frac{y_2}{y_1} \right)\).

Proof: Exercise (HW 3).

Geodesics.

Thm (Stahl 4.2.1): The geodesic segments of \(\mathbb{H}^2\) are

(i) Arcs of Euclidean semicircles centered on the \(x\)-axis
(ii) Segments of Euclidean vertical straight lines.

Proof: Let \(P = (x_1, y_1)\) and \(Q = (x_2, y_2)\) be two points in \(\mathbb{H}^2\), and let \(\Gamma\) be a curve joining them.

Case 1: \(x_1 \neq x_2\).

Let \(C\) be the point on the \(x\)-axis perpendicular bisector to \(PQ\).

Use polar coordinates, and let \(r = f(\theta)\):

\[
\begin{align*}
x &= c + r \cos \theta \\
y &= r \sin \theta
\end{align*}
\]

Calculus \(\Rightarrow\)

\[
\begin{align*}
\frac{dx}{d\theta} &= \frac{dr}{d\theta} \cos \theta + r \frac{d}{d\theta} \cos \theta = r' \cos \theta - r \sin \theta \\
\frac{dy}{d\theta} &= \frac{dr}{d\theta} \sin \theta + r \frac{d}{d\theta} \sin \theta = r' \sin \theta + r \cos \theta
\end{align*}
\]

\[dx^2 + dy^2 = (r' \cos \theta - r \sin \theta)^2 d\theta^2 + (r' \sin \theta + r \cos \theta)^2 d\theta^2
\]

\[= (r'^2 (\cos^2 \theta + \sin^2 \theta) - 2 r r' (\cos \theta \sin \theta - \sin \theta \cos \theta) + r^2 (\sin^2 \theta + \cos^2 \theta)) d\theta^2
\]

\[= (r'^2 + r^2) \ d\theta^2.
\]
Now, \[ |T| = \int_{\alpha}^{\beta} \frac{\sqrt{(r^2 + r^2 \sin^2 \theta)}}{r \sin \theta} \, d\theta = \int_{\alpha}^{\beta} \frac{\sqrt{r^2}}{r \sin \theta} \, d\theta = \int_{\alpha}^{\beta} \csc \theta \, d\theta = \ln \left( \frac{\csc \beta - \cot \beta}{\csc \alpha - \cot \alpha} \right) = |T_{pP}|, \]

where \( T_{pP} \) is the path from \( P \) to \( Q \) along the arc of the circle with center \( C \).

Since \( |T| \geq |T_{pP}| \) for any path \( T \), \( T_{pP} \) is a geodesic segment.

**Case 2:** \( x_1 = x_2 \).

Use rectangular coordinates, let \( x = F(y) \), so \( f' = \frac{dx}{dy} \).

\[ |T| = \int_{y_1}^{y_2} \sqrt{\left( f'(y) \right)^2 + 1} \, dy = \int_{y_1}^{y_2} \frac{\sqrt{\left( f'(y) \right)^2 + 1}}{y} \, dy = \ln \left( \frac{y_2}{y_1} \right) = |T_{pP}|, \]

where \( T_{pP} \) is the straight-line path from \( P \) to \( Q \).

Since \( |T| \geq |T_{pP}| \) for any path \( T \), \( T_{pP} \) is a geodesic segment.

We now know that hyperbolic geodesics are either

(a) Euclidean circle arcs, called **bounded geodesics**, or

(b) Euclidean vertical straight lines, called **straight geodesics**.

**Notation:** Let \( h(P, Q) \) denote the hyperbolic distance from \( P \) to \( Q \).

**Example:** Let \( A = (8, 4), \ B = (0, 8) \).

Then \( C = (1, 0) \) is the point on the \( x \)-axis equidistant to \( A \) and \( B \) (in \( \mathbb{E}^2 \)).

Note: \( \csc \beta = \frac{\sqrt{5}}{8}, \ \cot \beta = -\frac{1}{8}, \ \csc \alpha = \frac{\sqrt{65}}{7}, \ \cot \alpha = \frac{2}{7} \)

\[ \Rightarrow h(A, B) = |T_{AB}| = \ln \left( \frac{\sqrt{5}/8}{\sqrt{65}/7 - (2/7)} \right) \approx 1.450. \]
We didn't prove it rigorously (see Stahl, section 2.9), but Euclidean isometries have the following properties:

(i) They are generated by reflections

(ii) They are one of 3 types: (a) translation (b) rotation (c) glide-reflection

It turns out that (i) still holds for $H^2$, but (ii) does not.

For now, we'll primarily focus on hyperbolic reflections.

Thm (Stahl, 4.4.1): The following transformations of $H^2$ preserve both hyperbolic length and angle:

(a) inversions $I_{c,r}$, where $C$ is on the $x$-axis.

(b) reflections $P_m$, where $m$ is a line orthogonal to the $x$-axis.

(c) translations $T_{AB}$, where $AB$ is parallel to the $x$-axis.

Proof:

(a) Let $I_{c,r}$ be an inversion. By Thm 2.2.1, $I_{c,r}$ preserves angles.

Consider the curve $\gamma: [a,b] \to H^2$, where $r = F(\theta)$, $a \leq \theta \leq b$.

Then $I_{c,r}$ maps $\gamma$ to the curve $\gamma'$

given by $r = F'(\theta) = \frac{k^2}{F(\theta)}$, $a \leq \theta \leq b$. 
Now, \( |Γ'| = \int_α^β \frac{\sqrt{r'(t)^2 + r''(t)^2}}{r(t)} \, dt = \int_α^β \frac{\sqrt{r'(t)^2 + r''(t)^2}}{r(t)} \, dt \)

\[
= \int_α^β \frac{k^2 \sqrt{\left( \frac{r'(t)}{k} \right)^2 + \left( \frac{r''(t)}{k} \right)^2}}{k^2 \sin \theta / F} \, dt = \int_α^β \frac{\sqrt{\left( f'(t) \right)^2 + f''(t)^2}}{f'(t)} \, dt
\]

\[= \int_α^β \frac{\sqrt{f'(t)^2 + f''(t)^2}}{f'(t)} \, dt = |Γ'|. \checkmark \]

(b) Let \( m \) be the line \( x = c \).

2 points are symmetric w.r.t. \( m \) iff

\[ c = \frac{x_1 + x_2}{2}, \quad y_1 = y_2. \]

Let \( T(t) = (u(t), v(t)) \), \( a < t < b \) be a path.

Then \( p_m T(t) = (2c - u(t), v(t)) \), and

\[ T'(t) = (u'(t), v'(t)) \Rightarrow dx = u'(t) \, dt, \quad dy = v'(t) \, dt \]

\((p_m T)'(t) = (-u'(t), v'(t)) \Rightarrow dx = -u'(t) \, dt, \quad dy = v'(t) \, dt \)

Clearly, \( |p_m T| = \int_α^β \frac{\sqrt{u'(t)^2 + v'(t)^2}}{v'(t)} \, dt = |T'|. \checkmark \)

(C) Let \( T(x, y) = (x + h, y) \), fixed \( h \).

\[ T(t) = (u(t), v(t)), \quad a < t < b \]

\[ \Rightarrow T_T(t) = (u(t) + h, v(t)) \Rightarrow dx = u'(t) \, dt, \quad dy = v'(t) \, dt \]

\[ \Rightarrow |T_T| = \int_α^β \frac{\sqrt{u'(t)^2 + v'(t)^2}}{v'(t)} \, dt = |T'|. \checkmark \]

Cor. (Stahl 4.4.2): All of the transformations in Thm 4.4.1 carry geodesics to geodesics.