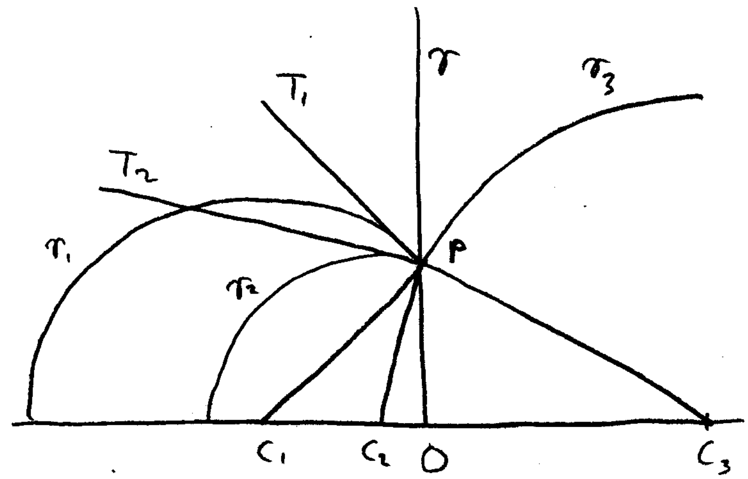
5. "Highschool" hyperbolic geometry

Prop (Stahl 6.1.6): Consider the following diagram, where γ_i has Euclidean center C_i and T_i is the tangent line at P.



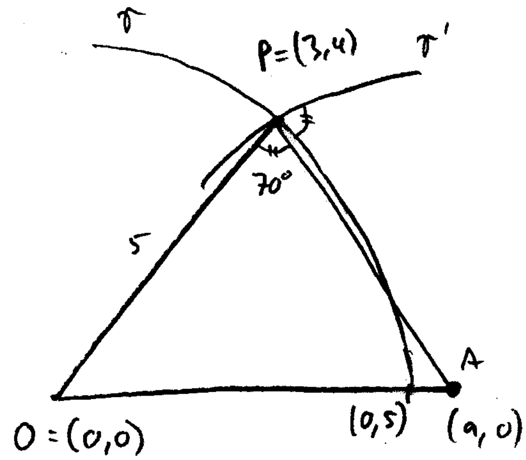
Then:

$$\measuredangle(\gamma, \gamma_1) = \measuredangle OC_1P \qquad \measuredangle(\gamma_1, \gamma_2) = \measuredangle C_1PC_2, \qquad \measuredangle(\gamma_3, \gamma_1) = \pi - \measuredangle C_1PC_3.$$

Proof: Exercise (HS geometry)

Applications!

Example (Stahl 6.1.2): Let $P = (3,4)$, and let γ be the bowed geodesic through P centered at $(0,0)$.



Goal: Construct a bowed geodesic γ' through P with angle $70°$ to γ.

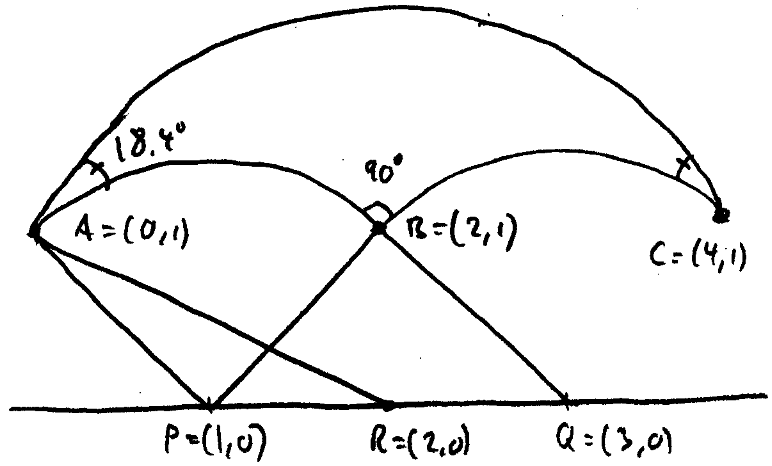
$$\tan(\measuredangle AOP) = \frac{4}{3} \Rightarrow \measuredangle AOP \approx 53.1 \Rightarrow \measuredangle PAO \approx 56.9°$$

Law of sines $\Rightarrow$ $$a = \frac{5 \sin 70°}{\sin(\measuredangle PAO)} \approx 5.6$$

$$AP = \frac{5 \sin(\measuredangle AOP)}{\sin(\measuredangle PAO)} \approx 4.8.$$

Now, $\measuredangle(\gamma', \gamma) = \measuredangle APO = 70°$.

<u>Example</u> (Stahl 6.1.3)



Consider the hyperbolic triangle through $A = (0,1)$, $B = (2,1)$ and $C = (4,1)$.

Clearly, $\measuredangle(AC, BC) = \measuredangle(AB, AC)$

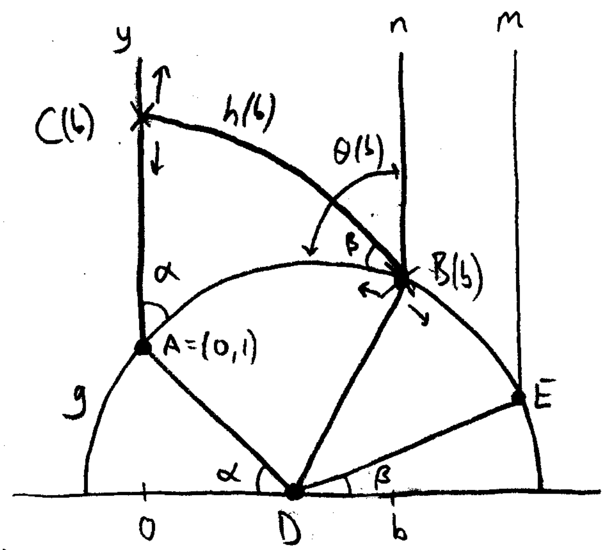
$= \measuredangle PAR$ (Prop 6.1.1)

$$= \cos^{-1}\left(\frac{AP^2 + AR^2 - PR^2}{2\, AP \cdot AR}\right) = \cos^{-1}\left(\frac{2+5-1}{2\sqrt{2}\sqrt{5}}\right) \approx 18.4°$$

Also by Prop 6.1.1, $\measuredangle(CB, AB) = \pi - \measuredangle PBQ = \pi - \cos^{-1}\left(\frac{BP^2 + BQ^2 - PQ^2}{2\, BP \cdot BQ}\right)$

$$= \pi - \cos^{-1}\left(\frac{2+2-4}{2\sqrt{2}\sqrt{2}}\right) = \pi - \cos^{-1}(0) = 90°.$$

<u>Theorem</u> (Stahl 6.1.4): Given any 3 angles whose sum is less than π, they are the angles of some hyperbolic triangle.



<u>Proof</u>: Suppose $\alpha + \beta + \gamma < \pi$ and WLOG assume $\alpha < \frac{\pi}{2}$.

Let $A = (0,1)$, $D = (d,0)$ s.t. $\measuredangle ADO = \alpha$.

Let g be the geodesic through A centered at D.

Let E be the point on g s.t. $\measuredangle(\text{x-axis}, DE) = \beta$.

Prop 6.1.1 $\Rightarrow$ $\measuredangle(g, \text{y-axis}) = \alpha$, $\measuredangle(m, g) = \beta$.

Pick any $B = (b, \cdot)$ on g between A & E.

<u>Note</u>: $\theta(b) := \measuredangle(n, g) = \measuredangle(\text{x-axis}, DB(b))$ is a monotonically decreasing function of b, and $\beta < \theta(b) < \pi - \alpha$.

Thus, for each β, there is a geodesic $h(b)$ to the y-axis such that $\measuredangle(h(b), g) = \beta$.

Let $C(b) = (0, c(b)) = h(b) \cap$ y-axis, and let $\gamma(b)$ be the corresponding angle. Note that $0 < \gamma(b) < \pi - a$ and γ is continuous.

Since $\pi - a > \gamma > 0$, $\exists\, b_0$ s.t. $\gamma(b_0) = \gamma$.

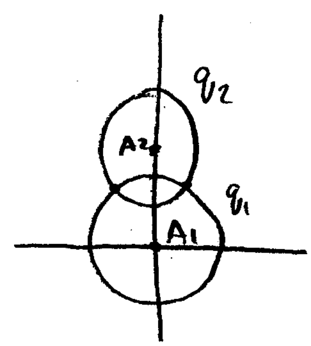
Triangle $A\,B(b_0)\,C(b_0)$ is our desired triangle. $\square$

<u>Next goal</u>: Prove the aforementioned "3 reflection theorem" in $\mathbb{H}^2$, with relying on Euclid's postulates.

We'll first prove some elementary results in $\mathbb{E}^2$, and then make a clever argument to extend them to $\mathbb{H}^2$.

<u>Theorem</u> (Hsu 3.1.1): For $i = 1, 2$ let $A_i \in \mathbb{E}^2$ and let q_i be the circle of center A_i and radius k_i. Let m be the line through A_1 & A_2. There are 3 possibilities:

(1) $q_1 \cap q_2 = \emptyset$
(2) $q_1 \cap q_2$ is a point on m.
(3) $q_1 \cap q_2 = \{B_1, B_2\}$ with $\rho_m(B_1) = B_2$.



<u>Proof</u>: WLOG assume standard position.

$A_1 = (0,0)$, $A_2 = (0,a)$, $m = y\text{-axis}$.

The circles are the points solving

$$q_1 : x^2 + y^2 = k_1^2$$
$$q_2 : x^2 + (y-a)^2 = k_2^2$$

Solve this system: $q_1 - q_2$: $\quad 2ay - a^2 = k_1^2 - k_2^2$

$$\Rightarrow \quad y = \frac{k_1^2 - k_2^2 + a^2}{2a} \quad \text{and} \quad x^2 = k_1^2 - y^2.$$

3 cases: (1) $k_1^2 - y^2 < 0$ no solutions for x

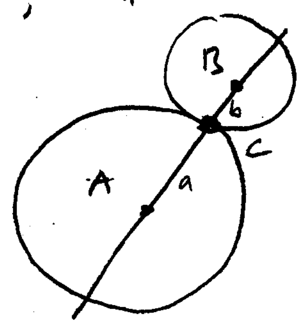
(2) $k_1^2 - y^2 = 0$ one solution: $x = 0$

(3) $k_1^2 - y^2 > 0$ $x = \pm\sqrt{k_1^2 - y^2}$ (2 solutions)

It's clear in this case that $P_m(B_1) = B_2$. $\square$

<u>Cor 3.1.2</u>: Let m be the line through $A, B \in \mathbb{E}^2$, and $C \in m$. Let $a = d(A,C)$, $b = d(B,C)$.

Then C is the only point at distance a from A and b from B.

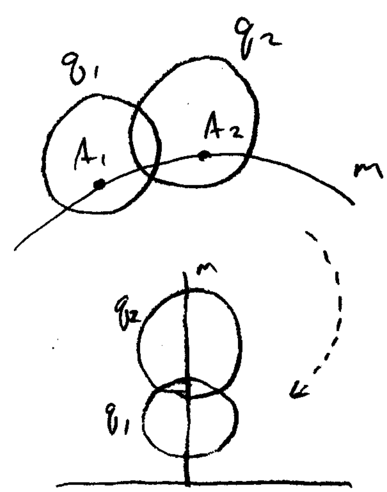


<u>Proof</u>: C is the intersection of circles with centers A & B, and radii a & b, and it's on m. Now apply Thm 3.1.1. $\square$

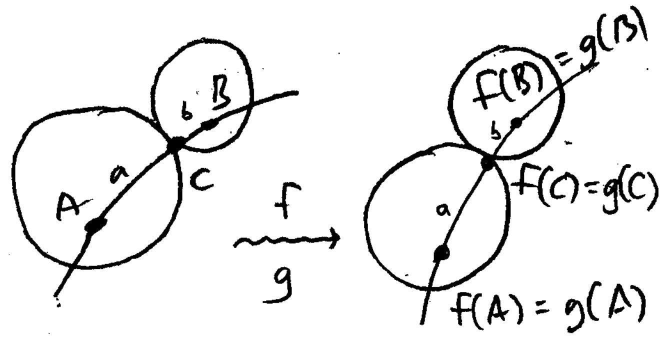
<u>Theorem</u> (Hsu 3.1.3): Theorem 3.1.1 holds in $\mathbb{H}^2$.

<u>Cor</u> (Hsu 3.1.4): Corollary 3.1.2 holds in $\mathbb{H}^2$.

<u>Proof</u>: WLOG use standard position, so m is a vertical geodesic. Since hyperbolic circles are Euclidean circles, the Thm. & Cor. now follow from the Euclidean case (Thm 3.1.1 & Cor 3.1.2). $\square$



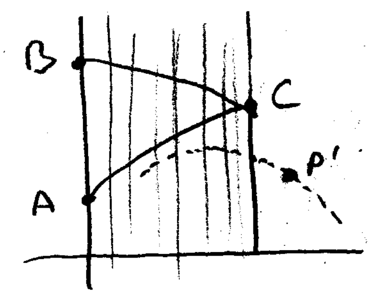
<u>Prop</u> (Hsu 3.2.1): If $f, g \in \mathrm{Isom}(\mathbb{H}^2)$ agree at points $A, B \in \mathbb{H}^2$, then they agree on the geodesic m through A & B.



<u>Proof</u>: Let $a = h(A, C)$, $b = h(B, C)$, where C is any point on m.

Since C is the unique point at distance a from A and b from B (Cor 3.1.4), $f(C)$ is the unique point at distance a from $f(A) = g(A)$ and distance b from $f(B) = g(B)$. Same is true for $g(C)$, and so by uniqueness, $f(C) = g(C)$. $\square$

<u>Theorem</u> (Hsu 3.2.2): Any 2 hyperbolic isometries f, g that agree at 3 noncolinear points are equal.



<u>Proof</u>: Let $A = (0,1)$, $B = (0,4)$, $C = (s,t)$ be 3 noncolinear points.

<u>Idea</u>: "Color in" all places where f & g agree.

By Prop 3.2.1, we can color all vertical geodesics s.t. $0 \le x \le s$.

Also, every point P not in this strip has a bowed geodesic from it into the strip, so we can color that geodesic, and P, as well. Thus f & g agree on all of $\mathbb{H}^2$. $\square$

<u>Remark</u>: The same proof works for $\mathbb{E}^2$ as well. Recall that this result was an "absolute" theorem.

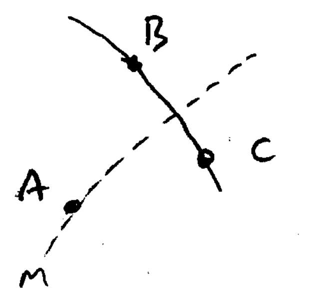
We can now prove the "3 reflections theorem."

The individual parts are left as an exercise (HW 4).

<u>Prop</u> (Hsu 3.3.1): Given any $A, B \in \mathbb{H}^2$, there is a hyperbolic reflection taking A to B.

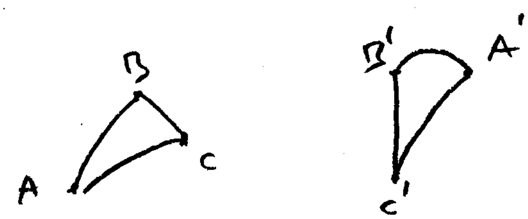
<u>Proof</u>: Exercise. (HW 4)

<u>Prop</u> (Hsu 3.3.2): Given $A \in \mathbb{H}^2$ and $B, C \in \mathbb{H}^2$ s.t. $h(A,B) = h(A,C)$, there is a hyperbolic reflection fixing A and taking B to C.



<u>Proof</u>: Exercise (HW 4)

<u>Theorem</u> (Hsu 3.3.4): Suppose A, B, C and A', B', C' are noncolinear points in $\mathbb{H}^2$ with $h(A,B) = h(A',B')$, $h(A,C) = h(A',C')$, & $h(B,C) = h(B',C')$. Then there exists at most 3 hyperbolic reflections whose composition takes A, B, C to A', B', and C', respectively.



<u>Proof</u>: Exercise (HW 4).

<u>Cor</u> (Hsu 3.3.5): Any hyperbolic isometry is the composition of at most 3 reflections. $\square$