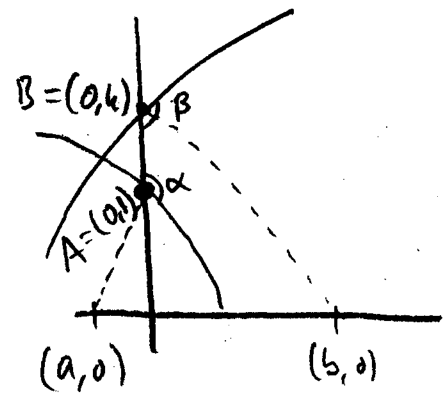
6. Hyperbolic triangles

The following is Euclid's 17th Proposition applied to $\mathbb{H}^2$:

Prop (Hsu 4.1.1): A hyperbolic triangle can have at most one non-acute angle.



Proof Suppose ΔABC has 2 non-acute angles $\alpha, \beta \geq \frac{\pi}{2}$.

WLOG, assume these are at points $A=(0,1)$ and $B=(0,k)$, and the triangle is in standard position (so $C=(c,0)$, $c>0$).

We'll show this is impossible.

The equations for the geodesics through A & B, respectively, are (see above & right)

$$(x-a)^2+y^2=a^2+1$$
$$(x-b)^2+y^2=b^2+k^2$$

By Prop 6.1.1, $b \geq 0 \geq a$.

Subtracting these equations yields $-2bx-2ax+b^2-a^2=k^2-1+b^2-a^2$

$\Rightarrow -2(b-a)x=k^2-1$.

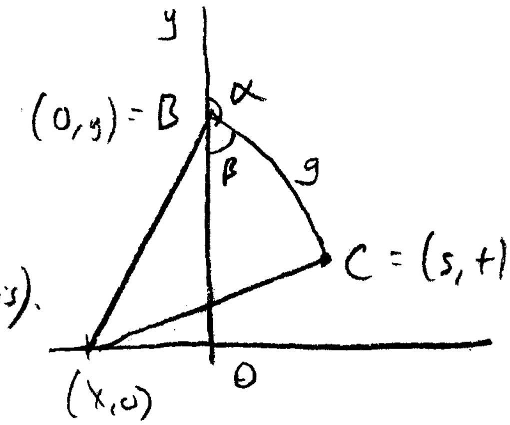
If $b-a>0$, $x=\dfrac{k^2-1}{-2(b-a)}<0$.

This is impossible, since $k>1$.

Thus, $a=b=0$, but this yields no solutions to the above system

2

Prop (Hsu 4.1.2): Fix $C=(s,t)\in\mathbb{H}^2$, $s>0$. Let g be a geodesic through C and the y-axis, with $B=(0,y)=g\cap(y\text{-axis})$.



Let $\alpha = yBC$, $\beta = OBC$.

Then (i) $y>\sqrt{s^2+t^2} \iff \alpha>\frac{\pi}{2} \iff \beta<\frac{\pi}{2}$

(ii) $y<\sqrt{s^2+t^2} \iff \alpha<\frac{\pi}{2} \iff \beta>\frac{\pi}{2}$

Proof: Let $(x,0)$ be the center of g as a Euclidean circle.

Since $(x,0)$ is equidistant from (s,t) and $(0,y)$,

$$x^2+y^2=(x-s)^2+t^2$$

This has a unique solution $x=\dfrac{s^2+t^2-y^2}{2s}$

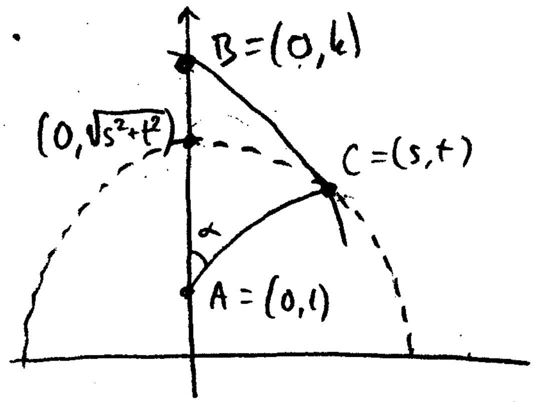
Now, $x<0 \iff \alpha>\frac{\pi}{2}$ and $x<0 \iff y>\sqrt{s^2+t^2}$. ✓

Case (ii) is proven similarly. □

Def: An altitude of ABC is a perpendicular dropped from a vertex to the geodesic through the other 2 vertices.

Prop (Hsu 4.2.2): Every hyperbolic triangle has an internal altitude.

Proof: By Prop 4.1.1, assume WLOG that $\alpha,\beta<\frac{\pi}{2}$ in triangle ABC, which is in standard position



The altitude from $C=(s,t)$, $s>0$ intersects the y-axis at $(0, \sqrt{s^2+t^2})$.

By Prop 4.1.2, $\alpha < \frac{\pi}{2} \Rightarrow \sqrt{s^2+t^2} > 1$

$\beta < \frac{\pi}{2} \Rightarrow \sqrt{s^2+t^2} > k$.

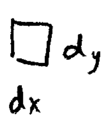
Thus ΔABC has an internal altitude. $\square$

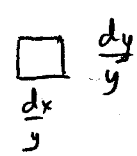
In $\mathbb{E}^2$, angles & area are independent. This is not true in $\mathbb{H}^2$.

Def: If R is a region in $\mathbb{H}^2$, define the hyperbolic area of R to be $ha(R) = \iint_R \frac{dx\,dy}{y^2}$.

Compare: In $\mathbb{E}^2$, area of R is $\iint_R dx\,dy$.

Reason:



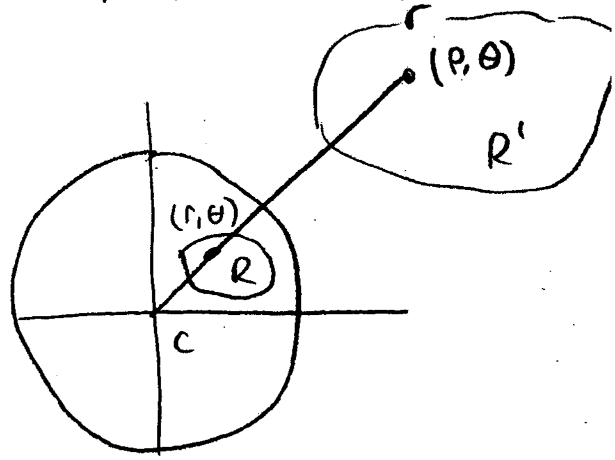


In $\mathbb{E}^2$, $dA = dx\,dy$ In $\mathbb{H}^2$, $dA = \frac{dx\,dy}{y^2}$

We must check: Hyperbolic area is preserved under hyperbolic isometries. It suffices to check circle inversion.

Consider the inversion $I_{C,k}(r,\theta) = \left(\frac{k^2}{r}, \theta\right)$.

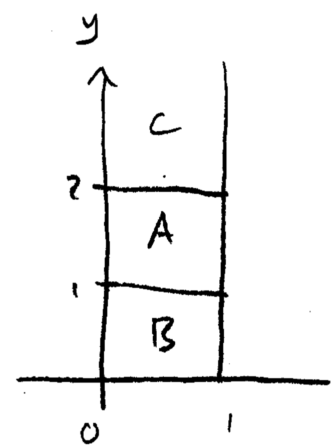
$$\frac{\partial(\rho,\theta)}{\partial(r,\theta)} = \begin{vmatrix} \rho_r & \rho_\theta \\ \theta_r & \theta_\theta \end{vmatrix} = \rho_r\theta_\theta - \rho_\theta\theta_r = -\frac{k^2}{r}\cdot 1 - 0\cdot 0 = -\frac{k^2}{r}$$



$$ha(R') = \iint_{R'} \frac{dx\,dy}{y^2} = \iint_{R'} \frac{\rho\,d\rho\,d\theta}{\rho^2 \sin^2\theta}$$

$$= \iint_R \frac{k^2/r}{\frac{k^4}{r^2}\sin^2\theta} \left|\frac{\partial(\rho,\theta)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= \iint_R \frac{r\,dr\,d\theta}{r^2\sin^2\theta} = \iint_R \frac{dx\,dy}{y^2} = ha(R).$$ ✓



<u>Examples:</u> (See diagram at right)

$$ha(A) = \int_1^2\int_0^1 \frac{dx\,dy}{y^2} = \int_1^2 \frac{dy}{y^2} = \frac{1}{2}$$

$$ha(B) = \int_0^1\int_0^1 \frac{dx\,dy}{y^2} = \int_0^1 \frac{dy}{y^2} = \infty$$

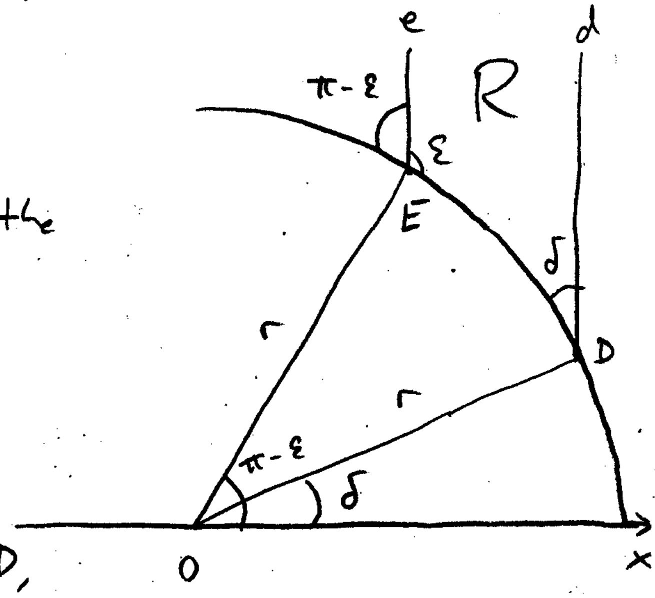
$$ha(C) = \int_2^\infty\int_0^1 \frac{dx\,dy}{y^2} = \int_2^\infty \frac{dy}{y^2} = \frac{1}{2}.$$

<u>Def:</u> The <u>defect</u> of a hyperbolic triangle with angles α, β, γ is $\pi - (\alpha+\beta+\gamma)$.

<u>Theorem</u> (Stahl 7.2.1) For any hyperbolic triangle ΔABC,

$$\text{area}(\Delta ABC) = \text{defect}(\Delta ABC)$$

First, we need a lemma.



<u>Lemma</u> (Stahl 7.2.2): Let DE be a bowed geodesic segment, and d, e the vertical rays through D & E. (See diagram at right). If R is the region between d & e and above DE, and $\delta = \angle dDE$, $\varepsilon = \angle eED$, then $ha(R) = \pi - \delta - \varepsilon$.

<u>Proof</u>: WLOG, assume DE is centered at O, radius r.

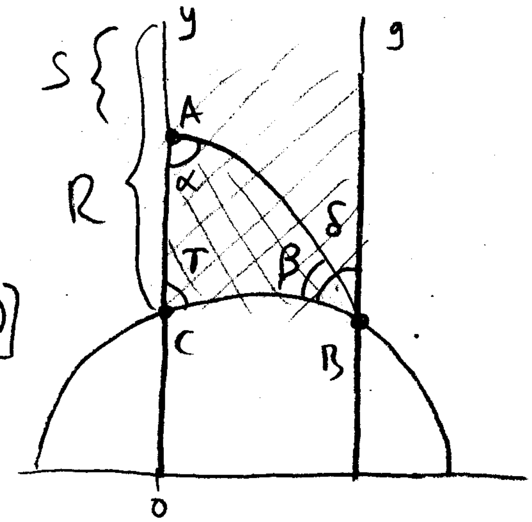
Prop 6.1.1 $\Rightarrow$ $\angle xOD = \delta$, $\angle xOE = \pi - \varepsilon$.

The bowed geodesic DE has equation $x^2 + y^2 = r^2$.

Now, $$ha(R) = \int_{r\cos(\pi-\varepsilon)}^{r\cos\delta} \int_{\sqrt{r^2-x^2}}^{\infty} \frac{dx\,dy}{y^2} = \int_{-r\cos\varepsilon}^{r\cos\delta} -y^{-1}\Big|_{\sqrt{r^2-x^2}}^{\infty} dx$$

$$= \int_{-r\cos\varepsilon}^{r\cos\delta} \frac{dx}{\sqrt{r^2-x^2}} = \sin^{-1}\left(\frac{x}{r}\right)\Bigg|_{-r\cos\varepsilon}^{r\cos\delta} = \sin^{-1}(\cos\delta) - \sin^{-1}(-\cos\varepsilon)$$

$$= \left(\frac{\pi}{2} - \delta\right) + \left(\frac{\pi}{2} - \varepsilon\right) = \pi - \delta - \varepsilon. \checkmark$$



<u>Proof</u> (Theorem 7.2.1).

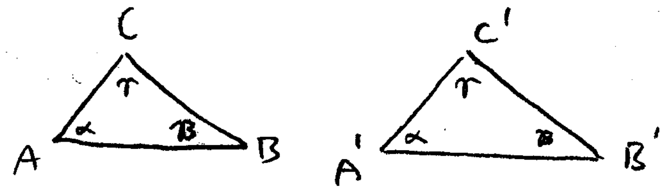
$$h(\Delta ABC) = ha(R) - ha(S)$$
$$= [\pi - \gamma - \angle(g, BC)] - [\pi - (\pi - \alpha) - \angle(g, AB)]$$
$$= \pi - \gamma - \alpha - [\angle(g, BC) - \angle(g, AB)]$$
$$= \pi - \alpha - \beta - \gamma.$$

$\square$

Cor: The sum of the angles of a hyperbolic triangle is less than π. [Converse to Thm 6.1.4 of Stahl.]

<u>Theorem</u> (Stahl 7.2.3): If the respective angles of two hyperbolic triangles are equal, then the triangles are congruent.

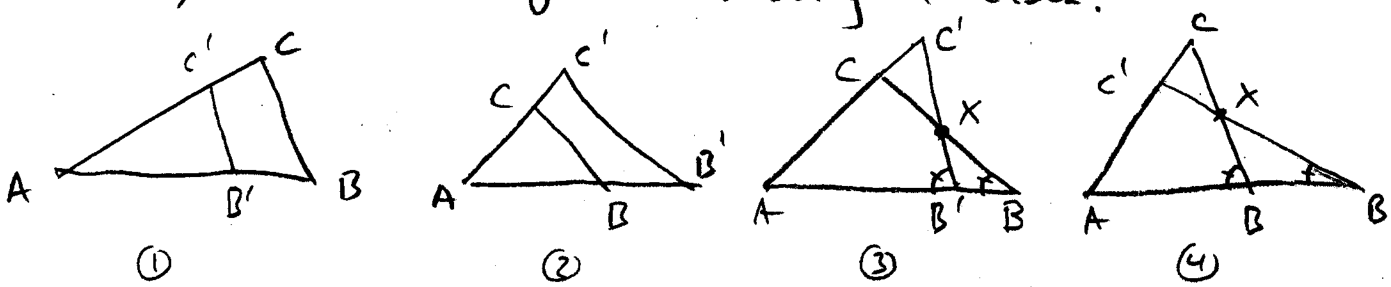
<u>Proof</u>: Consider two triangles:



Since all angles are congruent, we can <u>move</u> $A'B'C'$ by isometry so $A' = A$.

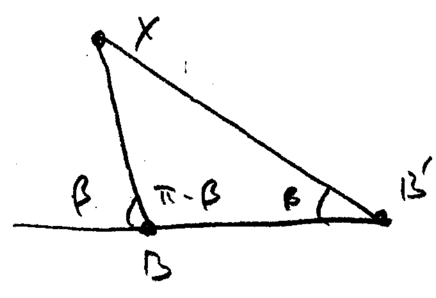
If $B' = B$ or $C' = C$, we're done (Euclid Prop 26, ASA).

Otherwise, we have one of the following 4 cases:



By Prop 7.2.1, $ha(ABC) = ha(AB'C')$ $\Rightarrow$ ① & ② are impossible.

But ③ & ④ have the following impossible situation: A triangle $\Delta BB'X$ with angles $\pi - \beta$ and β.



$\square$

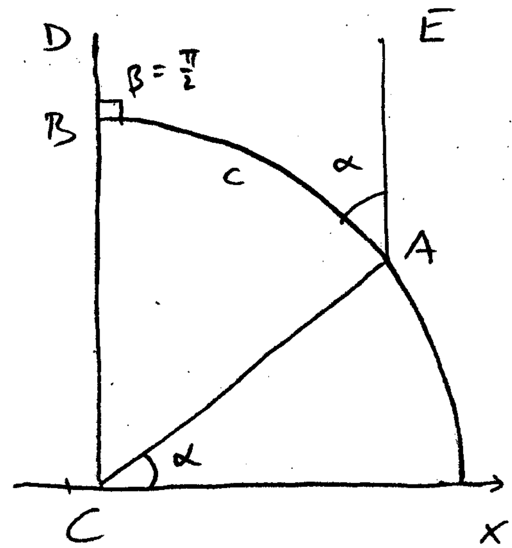
<u>Remark</u>. Theorem 7.2.3 implies that unlike $\mathbb{E}^2$, in $\mathbb{H}^2$, similar triangles $\Rightarrow$ congruent triangles. It means that $\mathbb{H}^2$ has an "AAA rule."

Prop (Stahl 8.1.1): Let AB be a bowed geodesic segment and let AE and BD be the straight geodesics above A and B. If $\alpha = \measuredangle EAB$, $\beta = \measuredangle ABD$, and $c = h(A,B)$, then

(i) $\sinh c = \dfrac{\cos\alpha + \cos\beta}{\sin\alpha \sin\beta}$

(ii) $\cosh c = \dfrac{1 + \cos\alpha \cos\beta}{\sin\alpha \sin\beta}$

(iii) $\tanh c = \dfrac{\cos\alpha + \cos\beta}{1 + \cos\alpha \cos\beta}$



Proof: First, consider the case $\beta = \frac{\pi}{2}$.

If C is the point on the x-axis below B, then by Stahl Prop 6.1.1, $\measuredangle xCA = \alpha$.

By Prop 4.1.1 of Stahl, $c = \ln \dfrac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha} = \ln \dfrac{\sin\alpha}{1-\cos\alpha}$

$$\Rightarrow e^c = \frac{\sin\alpha}{1-\cos\alpha}$$

$$\Rightarrow 2\sinh c = e^c - e^{-c} = \frac{\sin\alpha}{1-\cos\alpha} - \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin^2\alpha - 1 + 2\cos\alpha - \cos^2\alpha}{\sin\alpha(1-\cos\alpha)}$$

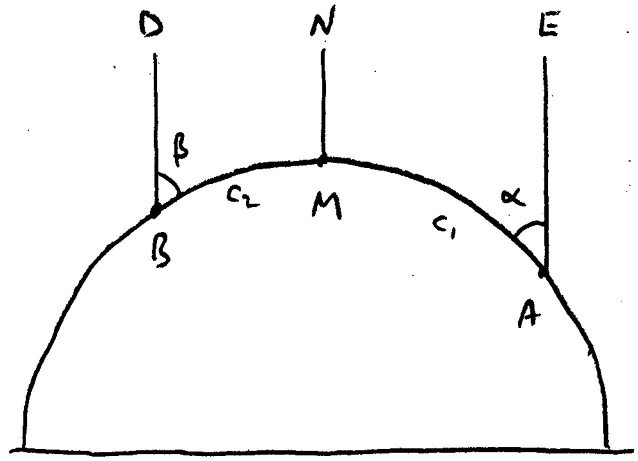
$$= \frac{2\cos\alpha(1-\cos\alpha)}{\sin\alpha(1-\cos\alpha)} = 2\cot\alpha$$

$\Rightarrow \sinh c = \cot\alpha$ when $\beta = \frac{\pi}{2}$ [thus, (i) holds]

Using the identies $\cosh^2 c - \sinh^2 c = 1$ and $\csc^2\alpha - \cot^2\alpha = 1$,

we get immediatly $\cosh c = \csc\alpha$, $\tanh c = \cos\alpha$ when $\beta = \frac{\pi}{2}$.

This confirms (ii) and (iii) when $\beta = \frac{\pi}{2}$.



Now consider the general case:

Let M be at the top of the geodesic through A & B.

Case (1) M is between A & B.

Let N be above M. Then $\measuredangle AMN = \measuredangle NMB = \frac{\pi}{2}$.

Let $c_1 = h(A,M)$, $c_2 = h(B,M)$, $c = c_1 + c_2$.

Now, $\sinh c = \sinh(c_1 + c_2) = \sinh c_1 \cosh c_2 + \cosh c_1 \sinh c_2$

$$= \cot\alpha \csc\beta + \csc\alpha \cot\beta = \frac{\cos\alpha + \cos\beta}{\sin\alpha \sin\beta}.$$

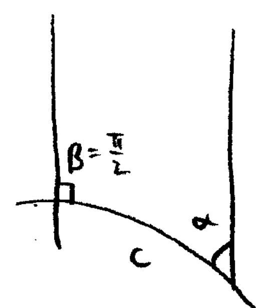
Case (2) M is outside of A & B. (Exercise).

This completes (i).

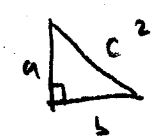
Parts (ii) and (iii) are left as an exercise.

Cor (Stahl 8.1.2): If $\beta = \frac{\pi}{2}$, then

$\sinh c = \cot\alpha$, $\cosh c = \csc\alpha$, $\tanh c = \cos\alpha$.



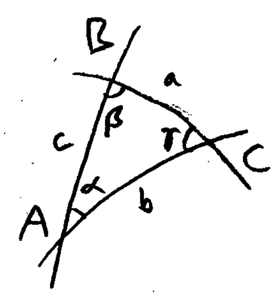
Recall: Euclidean Pythagorean Theorem: $a^2 + b^2 = c^2$



The following is the Hyperbolic Pythagorean Theorem.

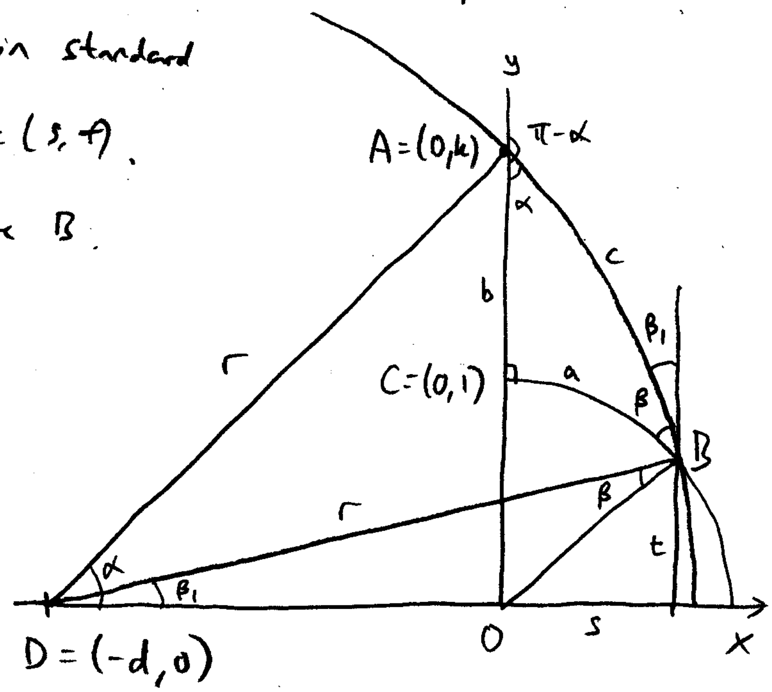
Theorem (Stahl 8.2.1): Let ABC be a hyperbolic triangle with a right angle at C. If a, b, c are the hyperbolic lengths opposite A, B and C, then

$$\cosh c = \cosh a \cosh b$$



Proof: WLOG assume ABC is in standard position: $A=(0,k)$, $B=(0,1)$, $C=(s,t)$.

Let $\beta_1 = \measuredangle$ between AB and line above B.



Prop 6.1.1 $\Rightarrow$ $\measuredangle ODA = \alpha$
$\measuredangle ODB = \beta_1$
$\measuredangle xOB = \beta + \beta_1$
$\measuredangle DBO = \beta$.

Cor 8.1.2 $\Rightarrow$ $\cosh a = \csc(\beta+\beta_1) = \frac{1}{t}$

Prop 4.1.3 $\Rightarrow$ $b = \ln k$ $\Rightarrow$ $\cosh b = \frac{e^b + e^{-b}}{2} = \frac{k + \frac{1}{k}}{2} = \frac{k^2+1}{2k}$.

Prop 8.1.1 $\Rightarrow$ $\cosh c = \frac{1 + \cos\beta_1 \cos(\pi - \alpha)}{\sin\beta_1 \sin(\pi-\alpha)} = \frac{1 - \frac{d+s}{r}\cdot\frac{d}{r}}{\frac{tk}{rr}} = \frac{r^2 - d^2 - ds}{kt}$

$$\Rightarrow \boxed{\cosh c = \frac{k^2 - ds}{kt}}$$

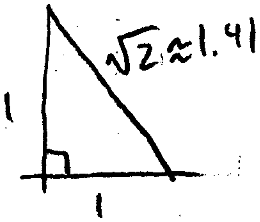
By Pythagorean theorem in $\mathbb{E}^2$: $s^2 + t^2 = 1$
$(s+d)^2 + t^2 = r^2 = d^2 + k^2$.

Thus, $2sd + d^2 = r^2 - 1$ $\Rightarrow$ $sd = \frac{r^2 - d^2 - 1}{2} = \frac{k^2-1}{2}$

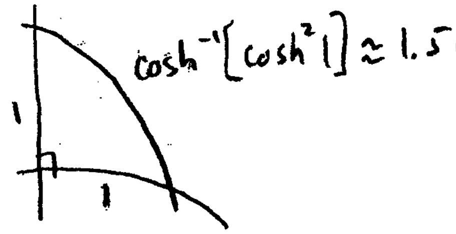
plug in for ds.

$$\Rightarrow \cosh c = \frac{k^2 - \frac{k^2-1}{2}}{2} = \frac{k^2+1}{2kt} = \cosh a \cosh b. \quad \square$$

Remark: Consider



$\mathbb{E}^2$



$\mathbb{H}^2$

If $a = b = 1$, $\gamma = \frac{\pi}{2}$, then $\cosh c = (\cosh 1)^2$

$$\Rightarrow \; c = \cosh^{-1}[\cosh^2 1] = \ln\left[\cosh^2 1 + \sqrt{[\cosh^2 1]^2 - 1}\right] \approx 1.51$$

Observation As $ha(\triangle ABC) \to 0$, $\alpha + \beta + \gamma \to \pi$, so as a triangle shrinks, it approaches a Euclidean triangle.

We'll show that the $\mathbb{H}^2$ Pythagorean Theorem $\longrightarrow$ $\mathbb{E}^2$ Pythagorean Theorem in this limit as well.

Since $e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \approx 1 + \frac{x^2}{2!}.$$

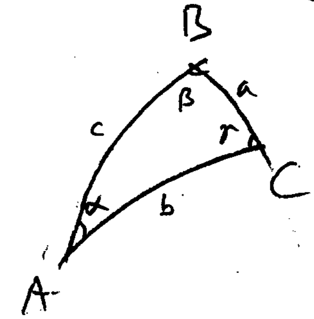
Thus, to a 3^{rd} order approximation,

$$\cosh c = \cosh a \cosh b \iff 1 + \frac{c^2}{2} = \left(1 + \frac{a^2}{2}\right)\left(1 + \frac{b^2}{2}\right)$$

$$1 + \frac{c^2}{2} = 1 + \frac{a^2}{2} + \frac{b^2}{2} + \underbrace{\cancel{\frac{a^2 b^2}{4}}}_{\approx 0}$$

$$\Rightarrow \; c^2 = a^2 + b^2.$$

There are many other trig identities for hyperbolic triangles, the proofs of which are left as Exercises.



__Prop__ (Stahl 8.2.2). Let ΔABC have angles α, β, γ as before. Then

(i) $\tanh a = \sinh b \tan\alpha$, $\tanh b = \sinh a \tan\beta$

(ii) $\sinh a = \sinh c \sin\alpha$, $\sinh b = \sinh c \sin\beta$

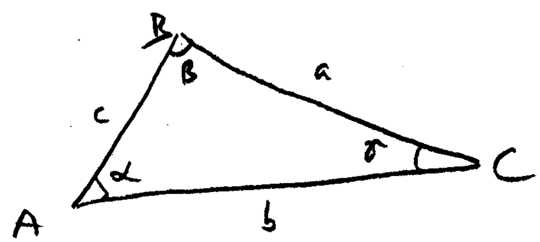
(iii) $\tanh b = \tanh c \cos\alpha$, $\tanh a = \tanh c \cos\beta$

(iv) $\cosh b \sin\alpha = \cos\beta$ $\cosh a \sin\beta \cos\alpha$

(v) $\cosh c = \cot\alpha \cot\beta$.

Recall the Euclidean __law of cosines__ and __law of sines__:

__Theorem__ (Stahl 8.3.1):



(i) $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

(ii) $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$.

The following (i and iii) is the hyperbolic version.

__Theorem__ (Stahl 8.3.2)

(i) $\cosh\alpha = \dfrac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}$ (ii) $\cosh a = \dfrac{\cos\beta\cos\gamma + \cos\alpha}{\sin\beta\sin\gamma}$.

(iii) $\dfrac{\sin\alpha}{\sinh a} = \dfrac{\sin\beta}{\sinh b} = \dfrac{\sin\gamma}{\sinh c}$.

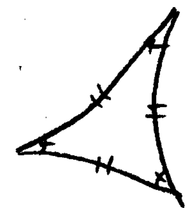
__Proof__: Very messy. See Stahl p. 105-106.

Examples:

(1) Consider an equilateral triangle with $\alpha = \beta = \gamma = \frac{\pi}{4}$.

Then $a = b = c$, and $\cosh a = \dfrac{\frac{1}{2} + \frac{\sqrt{2}}{2}}{\frac{1}{2}} = 1 + \sqrt{2}$

$\Rightarrow a = \cosh^{-1}(1+\sqrt{2}) \simeq 1.528$



(2) Consider an equilateral triangle with $a = b = c = 2$.

$\cosh 2 \approx 3.762$, $\sinh 2 = 3.627$

$$\Rightarrow \cos \alpha \approx \frac{(3.762)^2 - 3.762}{3.627} \approx 0.79$$

$\Rightarrow \alpha = \cos^{-1} 0.79 \approx 0.66$ radians

$\Rightarrow$ area $= \pi - (\alpha + \beta + \gamma) \approx 1.66$ radians.

Remark: Similar to the Pythagorean theorem, in the limit as $\alpha, \beta, \gamma \to 0$, the $\mathbb{H}^2$ laws of cosines & sines approach the $\mathbb{E}^2$ laws of cosines & sines.