7. The Euclidean isometry group

Recall that the set $\text{Isom}(E^2)$ of Euclidean isometries is a group, under function composition:

* If $f, g \in \text{Isom}(E^2)$, then $f \circ g \in \text{Isom}(E^2)$.
* $\text{Id} \in \text{Isom}(E^2)$ such that $f \circ \text{Id} = \text{Id} \circ f = f$ for each $f$.
* For each $f \in \text{Isom}(E^2)$, $f^{-1} \in \text{Isom}(E^2)$.
* Associative law holds.

Note that $\text{Isom}(E^2)$ is non-abelian, because in general, $f \circ g \neq g \circ f$.

We'll study the conjugacy classes of $\text{Isom}(E^2)$ next.

Def: Let $G$ be a group. We say that $f, g \in G$ are conjugate if there is some $h \in G$ such that $h f h^{-1} = g$.

Remark: Conjugacy is an equivalence relation. Elements in the same conjugacy class "have the same structure."

Example: In linear algebra, conjugacy is usually called similarity.

Let $G = \text{GL}_n(\mathbb{R})$, the set of invertible matrices.

$A \sim B$ iff $A$ & $B$ represent the same abstract linear transformation relative to different bases. [Change of basis theorem]
In linear algebra, there are certain properties (eigenvectors, determinant, trace, Jordan canonical form, etc.) that are invariant under change of basis, i.e., conjugacy.

Goal: Do the same thing for \( \text{Isom}(\mathbb{R}^3) \), and then \( \text{Isom}(H^3) \).
Throughout, let \( X \) be a geometry and \( G = \text{Isom}(X) \).

Theorem (Hrn 5.1.3): For \( f, g \in G \), \( x, y \in X \), \( g \) takes \( x \) to \( y \)
iff \( fgf^{-1} \) takes \( f(x) \) to \( f(y) \)

Proof: Exercise.

Def: For \( g \in G \), the fixed points of \( g \) are the points \( x \in X \) such that \( g(x) = x \).

Cor: (Hrn 5.1.5) \( x \in X \) is a fixed point of \( g \) iff \( f(x) \) is a fixed point of \( fgf^{-1} \).

Remark: This holds more generally for any group \( G \) that acts on a set \( X \).

Let \( f: \mathbb{R}^3 \to \mathbb{R} \) be a non-negative function. Recall that the infimum of \( f \), denoted \( \inf_{x \in \mathbb{R}^3} f(x) \), is the greatest lower bound on \( f(x) \). It exists, though \( f(x) \) might not ever achieve this bound. (It might get arbitrarily close.)
Def: Let $g \in \text{Isom}(\mathbb{E}^2)$. The minimal motion of $g$, is defined as $\mu(g) = \inf_{x \in \mathbb{E}^2} d(x, g(x))$. The set of minimal motion of $g$ is $\{ x \in \mathbb{E}^2 : d(x, g(x)) = \mu(g) \}$.

Theorem (Hsu 5.2.3): Let $f, g \in \text{Isom}(\mathbb{E}^2)$, let $S$ be the set of minimal motion of $g$, and $S'$ the set of minimal motion of $fg^{-1}$. Then $\mu(g) = \mu(fg^{-1})$ and $f(S) = S'$.

Proof: By def'n, $\mu(g) = \inf_{x \in \mathbb{E}^2} d(x, g(x))$

$= \inf_{x \in \mathbb{E}^2} d(f^{-1}(x), g(f^{-1}(x)))$

$= \inf_{x \in \mathbb{E}^2} d(f(f^{-1}(x)), f(g(f^{-1}(x))))$

$= \inf_{x \in \mathbb{E}^2} d(x, fg^{-1}(x)) = \mu(fg^{-1})$

Now, let $\mu = \mu(g) = \mu(fg^{-1})$.

Note that $x \in S \iff d(x, g(x)) = \mu \iff d(f(x), f(g(x))) = \mu \iff d(f(x), fg^{-1}(f(x))) = \mu \iff f(x) \in S'$.

Big idea: Theorem 5.2.3 says that the minimal motion of an isometry is an invariant of a conjugacy class.

Next goal: Determine the conjugacy classes of $\text{Isom}(\mathbb{E}^2)$. 
Approach:

1. Determine the minimal motion \(\mu\), set of minimal motion for each type of isometry.

2. Given an arbitrary isometry of a certain type, conjugate it to an isometry of the same type in standard position.

3. Find the conjugacy classes of each type by considering the ones in standard position.

Theorem (Hsu 5.3.1): The 4 types of Euclidean isometries have minimal motions \(\mu\), set of minimal motions as follows:

1. Reflection: \([P_m]\) has line \(m\) as its fixed point set (and so \(\mu(P_m) = 0\) and the set of minimal motion is \(m\)).

2. Rotation: \([R_{C,\alpha}]\) has one fixed point, \(C\). (So \(\mu(R_{C,\alpha}) = 0\) and the set of minimal motion is \(C\).)

3. Translation: \([T_v]\) has minimal motion \(\mu(T_v) = \{v\}\) and the set of minimal motion is \(E^2\).

4. Glide reflection: \([T_{AB}]\) has minimal motion \(\mu(T_{AB}) = d(A,B)\) and the set of minimal motion is the line through \(A-B\).

Cor: Reflections are only conjugate to reflections, rotations are only conjugate to rotations, and so on.
Theorem (Hsu 5.3.2): All Euclidean reflections are conjugate.

Proof: Let $P_k, P_m$ be reflections and pick any $f \in \text{Isom}(\mathbb{E}^2)$ that sends $m \mapsto k$. By Cor 5.1.5, $f_pm f^{-1}$ fixes every point on $k$. But the only non-trivial isometry fixing all points on $k$ is $P_k$, and so $f_pm f^{-1} = P_k$.

Classifying rotations by conjugacy takes more work.

Theorem: (Hsu 5.3.3) Two rotations $R_{c,\alpha}$ and $R_{d,\beta}$ with $-\pi < \alpha, \beta < \pi$ are conjugate $|d| = |b|$

Proof: Assume all angles are in $[-\pi, \pi]$.

Let $R_{c,\alpha}$ be a rotation and let $P \in \mathbb{E}^2$ be a point with $|C-P| = 1$ and let $d = d(P, R_{c,\alpha}(P))$.

By the law of cosines, $d = \sqrt{2 - 2 \cos \alpha}$.

Since $P$ is arbitrary, any rotation $R_{c,\alpha}$ moves all points at distance 1 from $C$ exactly $\sqrt{2 - 2 \cos \alpha}$.

Claim: Every conjugate of $R_{c,\alpha}$ is a rotation of angle $\pm \alpha$.

Pick $f \in \text{Isom}(\mathbb{E}^2)$ and let $D = f(C)$.

By Cor 5.1.5, $f R_{c,\alpha} f^{-1}$ fixes only $D$. 

\[ R_{c,\alpha}(P) \] 

\[ d \] 

\[ \alpha \] 

\[ P \] 

\[ C \] 

\[ l \] 

\[ P \] 

\[ C \] 

\[ d \] 

\[ \alpha \]
Thus, \( f \circ R_c,\alpha \circ f^{-1} \) is a rotation around \( D \); call it \( R_{D,\beta} \). Goal: Show \( \beta = \pm \alpha \).

Let \( x \in \mathbb{E}^2 \) be a point with \( |x-D| = 1 \).

Now, \( \sqrt{2 - 2 \cos \beta} = d(x, R_{D,\beta}(x)) = d(x, f \circ R_c,\alpha \circ f^{-1}(x)) = d(f^{-1}(x), R_c,\alpha \circ f^{-1}(x)) = \sqrt{2 - 2 \cos \alpha} \).

Therefore, \( \sqrt{2 - 2 \cos \beta} = \sqrt{2 - 2 \cos \alpha} \Rightarrow \cos \alpha = \cos \beta \Rightarrow \alpha = \pm \beta \).

It now remains to show that \( R_{c,\alpha} \) is conjugate to both \( R_{0,\beta} \).

2 cases: (i) \( R_{0,\alpha} \) (HW 1)

(ii) \( R_{0,-\alpha} \) (HW 6)

Now, we'll consider isometries without fixed points.

**Theorem (Hsu 5.3.4):** Two translations \( \tau_v \) and \( \tau_w \) are conjugate iff \( |v| = |w| \).

**Proof:**

\( (\Rightarrow) \) If \( \tau_v \) and \( \tau_w \) are conjugate, then by Thm 5.2.3,

\[ |v| = \tau_v = \tau_w = |w| \]

\( (\Leftarrow) \) Claim: If \( R \) is a rotation, then \( \tau_{R(v)} \) is conjugate to \( \tau_v \). (Exercise).

Now, if \( |v| = |w| \), let \( R \) be a rotation about \( 0 \) such that \( R(v) = w \). \( \Box \)
Theorem (Hsu 5.3.5): Two glide reflections $T_{AB}$, $T_{CD}$ are conjugate iff $d(A, B) = d(C, D)$.

Proof: Exercise. (HW 6)

Euclidean isometries, analytically

We can describe $f \in \text{Isom}(\mathbb{R}^2)$ as a bijection $f: \mathbb{C} \to \mathbb{C}$:

- **Translations**: $T_c(z) = z + c$
- **Rotations**: $R_{c, \alpha}(z) = e^{i\alpha}(z - c) + c$
- **Reflections**: $R_{m}(z) = e^{2i\alpha} \overline{z - c} + c$

Summary (Stein Thm 9.1.7): The isometries of $\mathbb{R}^2$ have the form $f(z) = e^{i\alpha} (z + c)$ (rotations, translations), or $f(z) = e^{i\alpha} \overline{z} + c$ (reflections, glide-reflections).

Moreover, all maps of these forms are isometries.

Proof: Exercise (elementary verification).