# 8. Möbius transformations & the hyperbolic isometry group

Goal: Classify hyperbolic isometries both analytically and algebraically (i.e, as quotients of matrix groups.)

We've already seen some hyperbolic isometries,

* Horizontal translation: $f(z) = z + r$
* Reflection in vertical line: $f(z) = -\bar{z} + r$
* Inversion $I_{0,k}$ : $f(z) = \frac{k^2}{\bar{z}}$
* Inversion $I_{c,k} = \tau_c \circ I_{0,k} \circ \tau_c^{-1}$: $f(z) = \frac{k^2}{\bar{z} - c} + c$

## Composing two reflections

Examples: 1

(i) $I_{0,2} : z \longmapsto \frac{4}{\bar{z}}$

$$1+i \longmapsto \frac{4}{\overline{1+i}} = \frac{4}{1-i} \cdot \frac{1+i}{1+i} = \frac{4+4i}{2} = 2+2i$$

(ii) Let $A = (3,0)$. $I_{A,4} : z \longmapsto \frac{4^2}{\bar{z}-3} + 3 = \frac{16}{\bar{z}-3} + \frac{3(\bar{z}-3)}{\bar{z}-3} = \frac{3\bar{z}+7}{\bar{z}-3}$.

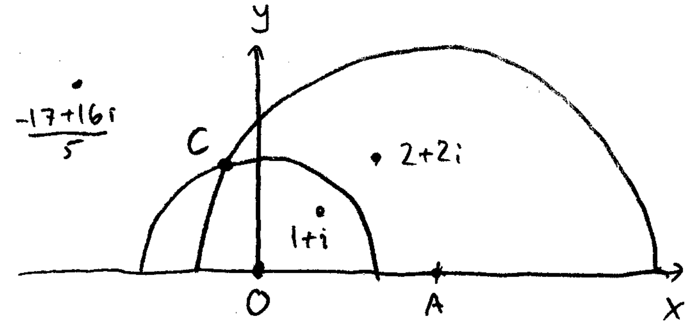
$$1+i \longmapsto \frac{3(\overline{1+i}) + 7}{\overline{1+i} - 3} = \frac{10-3i}{-2-i} \, \frac{-2+i}{(-2+i)} = \frac{-17+16i}{5}$$

Note: The axes of these two reflections intersect.

2

The composition $R := I_{A,4} \circ I_{0,2}$

is a "hyperbolic rotation"




(see the "absolute proposition",

Prop 2.2.6 in Stahl: $\rho_n \circ \rho_m = R_{A,2\alpha}$, where $\alpha = \measuredangle(n,m)$.)

$$R(z) = I_{A,4} \circ I_{0,2}(z) = I_{A,4}\left(\frac{4}{z}\right) = \frac{3\frac{4}{z}+7}{\frac{4}{z}-3} = \frac{7z+12}{-3z+4}.$$

The fixed point is found by solving $\dfrac{7z+12}{-3z+4} = z$

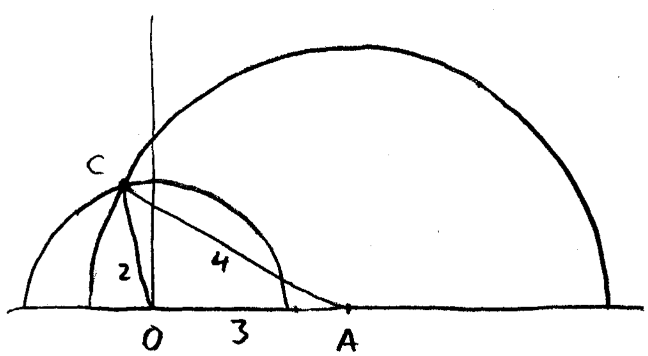
$$\Rightarrow 7z+12 = -3z^2+4z \quad \Rightarrow z^2+z+4=0 \quad \Rightarrow z = \frac{-1 \pm i\sqrt{15}}{2}.$$

Only $\dfrac{-1+i\sqrt{15}}{2}$ lies in the upper half-plane.

The angle of rotation is $2\alpha$,

where $\alpha = \cos^{-1}\left(\dfrac{AC^2+OC^2-OA^2}{2\cdot AC\cdot OC}\right)$

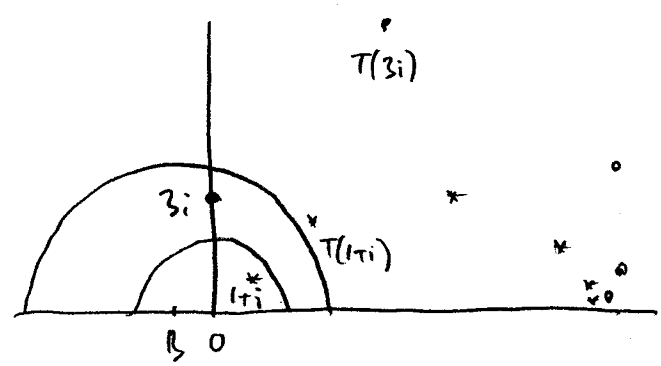
$$= \cos^{-1}\left(\frac{4^2+2^2-3^2}{2\cdot 4\cdot 2}\right) = \cos^{-1}\left(\frac{11}{16}\right),$$




so $\theta = 2\alpha \approx 93.13°$.

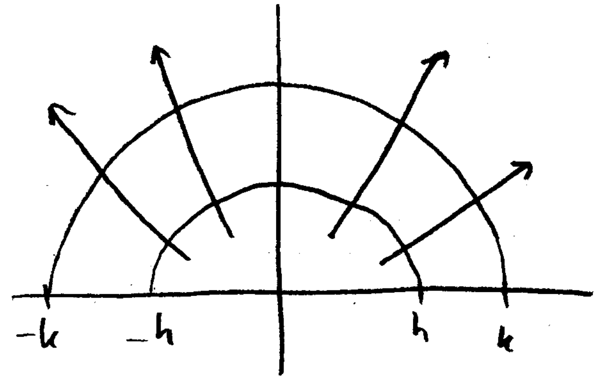
2 Consider $T = I_{B,4} \circ I_{0,2}$, where $B = (-1,0)$

Since the axes of reflection don't intersect, this is in some sense a "hyperbolic translation."



$$T(z) = I_{-1,4} \circ I_{0,2}(z) = \frac{4^2}{\frac{2^2}{z}+1} - 1 = \frac{15z-4}{z+4}$$

③ $$I_{0,k} \circ I_{0,h}(z) = \frac{k^2}{\overline{\left(\frac{h^2}{\bar z}\right)}} = \frac{k^2}{h^2} z$$




This is a <u>dialation</u>.

Thus, for any $\kappa > 0$, the dialation $D_\kappa(z) := \kappa z$ is a hyperbolic isometry!

<u>Theorem</u> (Stahl 9.2.4). The isometries of $\mathbb{H}^2$ are precisely those functions $\mathbb{C} \to \mathbb{C}$ that can be written like

(i) $f(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, or (ii) $f(z) = \dfrac{\alpha(-\bar z) + \beta}{\gamma(-\bar z) + \delta}$,

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ satisfy $\alpha\delta - \beta\gamma > 0$.

We'll actually state and prove a stronger more algebraic (and elegant!) version of this.

<u>Examples</u> of hyperbolic isometries in this form:

* Horizontal translations: $z \longmapsto \dfrac{1z + r}{0z + 1}$
* Reflections across vertical geodesic: $z \longmapsto \dfrac{1(-\bar z) + r}{0(-\bar z) + 1}$
* Reflections across bowed geodesic: $z \longmapsto \dfrac{k^2}{\bar z - a} + a = \dfrac{a\bar z + k^2 - a^2}{\bar z - a} = \dfrac{-a(-\bar z) + (k^2 - a^2)}{-(-\bar z) - a}$

Throughout, $F$ will be the field $\mathbb{R}$ or $\mathbb{C}$, $F^{\times} := F \setminus \{0\}$, and $\hat{F} := F \cup \{\infty\}$.

Def: The <u>general linear group</u>, denoted $GL_n(\mathbb{R})$, is the set of $n \times n$ invertible matrices. It is a <u>group</u> under multiplication.

The <u>center</u> of $GL_n(\mathbb{R})$, denoted $Z(GL_n(\mathbb{R}))$, is the subgroup of elements that commute with everything. These are just the scalars of the identity matrix. ~~Th~~

That is:

$$Z(GL_n(\mathbb{R})) := \{ A \in GL_n(\mathbb{R}) : AB = BA \text{ for all } B \in GL_n(\mathbb{R}) \}$$
$$= \{ cI : c \in \mathbb{R}^{\times} \}.$$

It is well-known that $Z(GL_n(\mathbb{R})) \triangleleft GL_n(\mathbb{R})$, which means the the <u>quotient group</u> $GL_n(\mathbb{R}) / Z(GL_n(\mathbb{R}))$ is a well-defined group. This group is called the <u>projective linear group</u>, and denoted $PGL_n(\mathbb{R})$.

It consists of the set of invertible $n \times n$ matrices under the equivalence that $A \sim B$ iff $B = cA$ for some $c \in \mathbb{R}^{\times}$; i.e, the elements of $GL_n(\mathbb{R})$ up to scalar multiple.

<u>Def:</u> An $n \times n$ <u>real elementary matrix</u> is one of the following transformations in the column vectors of $\mathbb{R}^n$; $i \neq j$

1. Exchanging the $i^{th}$ & $j^{th}$ coord: $[e_1 \cdots e_{\tau(i)} \cdots e_{\tau(j)} \cdots e_n]$
   Here: $\tau = (i\ j) \in S_n$.
2. Adding $a*$(column $i$) to column $j$: $(a \neq 0)$ $[e_1 \cdots e_{j-1}\ (e_j + a e_i)\ e_{j+1} \cdots e_n]$
3. Multiplying column $i$ by $a \neq 0$: $[e_1 \cdots e_{i-1}\ a e_i\ e_{i+1} \cdots e_n]$

<u>Remarks</u>:

* Elementary matrices are invertible, as they correspond precisely to "elementary column operations" that put a matrix in reduced row echelon form (RREF).
* The inverse of an elementary matrix is an elem. matrix.
* left-multiplication by elementary matrices corresponds to elementary <u>row</u> operations.

<u>Example</u>: The $2 \times 2$ elementary matrices are (for $a \in \mathbb{R}^*$):

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

<u>Theorem</u> (Hsu 6.1.3): The $n \times n$ real elementary matrices generate $GL_n(\mathbb{R})$.

Proof (sketch): Every invertible $n \times n$ matrix may be row-reduced to the identity matrix.

Algebraically, there is a sequence $E_1, \dots, E_k$ of elementary matrices such that $E_k \cdots E_2 E_1 A = I$

$$\Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}. \qquad \square$$

Remark: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$

and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$.

Cor (Hsu 6.1.4): $GL_n(\mathbb{R})$ is generated by the matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ for all $a \neq 0$.

Recall: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ and $\operatorname{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$

Also, $\det(A - \lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A$

$\det(AB) = (\det A)(\det B)$.

Theorem (Hsu 6.1.5): $\det(PAP^{-1}) = \det A$, $\operatorname{tr}(PAP^{-1}) = \operatorname{tr} A$, i.e., determinant & trace are conjugacy class invariants in $GL_2(\mathbb{R})$.

Remark: If $A \in GL_2(\mathbb{R})$, then $\det(kA) = k^2(\det A)$, $\operatorname{tr}(kA) = k(\operatorname{tr} A)$.

Therefore, det & tr are not well-defined notions in $PGL_2(\mathbb{R})$, i.e, equivalent elements may have different values.

We can fix this: define $\tau(A) = \dfrac{(\operatorname{tr} A)^2}{\det A}$.

Note that $\tau(kA) = \tau(A)$, so $\tau$ is well-defined in $PGL_2(\mathbb{R})$.

By Theorem 6.1.5, we have:

Cor 6.1.6: $\tau$ is a conjugacy class invariant of $PGL_2(\mathbb{R})$.

Def. The Möbius map $\mu$ is the function that sends $A \in GL_2(\mathbb{R})$ to the function $\mu_A$, where

$$\mu_A(z) = \begin{cases} \dfrac{az+b}{cz+d} & \det A > 0 \\ \dfrac{a\bar{z}+b}{c\bar{z}+d} & \det A < 0 \end{cases}$$

Concern: The function isn't defined for all of $\mathbb{C}$, because the denominator could be zero.

We can fix this too.

$\boxed{8}$

Let $Z = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} : z_i \in \mathbb{C} \right\}$

Consider the "bar" map: $Z \longrightarrow Z$,

$$\overline{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}} \longmapsto \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}.$$

<u>Observations</u>: $\ast\ \overline{\overline{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, i.e., $\overline{\phantom{x}}^{2} = \mathrm{id}$.

$\ast\ \overline{Az} = A\bar{z}$ if $A \in GL_2(\mathbb{R})$.

Define an <u>action</u> of $GL_2(\mathbb{R})$ on $Z$ by the rule

$$A.z = \begin{cases} Az & \text{if } \det A > 0 \\ A\bar{z} & \text{if } \det A < 0. \end{cases}$$

We must check that $(AB).z = A.(B.z)$.

There are <u>4 cases</u>:

$\det A, \det B > 0$: $(AB).z = ABz = A.(B.z)$ ✓

$\det A > 0, \det B < 0$: $(AB).z = AB\bar{z} = A.(B.z)$ ✓

$\det A < 0, \det B > 0$: $(AB).z = AB\bar{z} = A\overline{Bz} = A.(B.z)$ ✓

$\det A, \det B < 0$: $(AB).z = ABz = AB\bar{\bar{z}} = A\overline{B\bar{z}} = A.(B.z)$ ✓

Remark: The action $A.z$ preserves the "nonzero scalar multiple" equivalence on $Z$:

$$\text{Take } A \in GL_2(\mathbb{R}), \quad \lambda \in \mathbb{C}^\times, \quad z \in Z.$$

$$A\lambda z = \lambda A z \quad \text{and} \quad \overline{\lambda z} = \bar{\lambda}\bar{z}, \quad \text{and so}$$

$$A.(\lambda z) = \lambda'(A.z) \quad \text{for some } \lambda' \in \mathbb{C}.$$

Thus, this action carries over to an action on $Z/\sim$.

Key idea: There is a natural bijection

$$\Phi : Z/\sim \longrightarrow \hat{\mathbb{C}}, \qquad \Phi : \lambda\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{cases} z_1/z_2 & z_2 \neq 0 \\ \infty & z_2 = 0. \end{cases}$$

This is well-defined because $\dfrac{z_1}{z_2} = \dfrac{\lambda z_1}{\lambda z_2}$.

We still need to show (HW):

1. $\Phi$ is a bijection
2. Given the conventions that $\dfrac{a\infty + b}{c\infty + d} = \dfrac{a}{c}$ and $\dfrac{a}{0} = \infty$, then for $z \in Z/\sim$, we have $\mu_A(\Phi(z)) = \Phi(A.z)$.

Another way to see this:

Recall that an action of a group $G$ on a set $X$ is a homomorphism $\varphi : G \longrightarrow S_X$.

$$\begin{tikzcd} Z/\sim \arrow[r, "\text{action by } A"] \arrow[d, "\Phi"'] & Z/\sim \arrow[d, "\Phi"] \\ \hat{\mathbb{C}} \arrow[r, "\text{action by } A"] & \hat{\mathbb{C}} \end{tikzcd}$$

**Theorem** (Hsu 6.2.2): Given the conventions that $\frac{a\infty+b}{c\infty+d} = \frac{a}{c}$ and $\frac{a}{0} = \infty$, the Möbius map is a homomorphism $\mu : GL_2(\mathbb{R}) \longrightarrow S_{\hat{\mathbb{C}}}$ (set of permutations of $\hat{\mathbb{C}}$). In other words, $\mu$ defines an action of $GL_2(\mathbb{R})$ on $\hat{\mathbb{C}}$.

**Proof**: To show that $\mu$ is indeed a homomorphism, we must show that for any $z \in \mathbb{Z}/\sim$, $\mu_{AB}(\Phi(z)) = \mu_A \mu_B(\Phi(z))$.

This holds because

$$\mu_{AB}(\Phi(z)) = \Phi(AB.z) = \Phi(A.(B.z)) = \mu_A \Phi(B.z) = \mu_A\mu_B(\Phi(z)). \quad \square$$

**Theorem** (Hsu 6.2.3): The Möbius map is a surjective (onto) homomorphism $\mu : GL_2(\mathbb{R}) \longrightarrow \mathrm{Isom}(\mathbb{H}^2) \leq S_{\hat{\mathbb{C}}}$

**Proof**: We need to show 2 things:

(1) Show that for any $A \in GL_2(\mathbb{R})$, $\mu_A \in \mathrm{Isom}(\mathbb{H}^2)$.

(2) Show that $\mu$ maps onto $\mathrm{Isom}(\mathbb{H}^2)$.

Step 1: Since $\mu$ is a homomorphism, it suffices to prove (1) just for the generators of $GL_2(\mathbb{R})$.

(i) $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. $\mu_A(z) = z + a$ is a horizontal translation ✓

(ii) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\qquad \mu_A(z) = \frac{1}{\bar{z}}$ is the inversion $I_{0,1}$ ✓

(iii) $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ $a > 0$. $\qquad A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\Rightarrow \mu_A = I_{0,1} \circ I_{0,\sqrt{a}}$$ ✓

(iv) $A = \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}$ $a < 0$ $\qquad A = \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Note that $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the reflection in the vertical line $x=0$

and $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \xmapsto{\mu_A} I_{0,1} \circ I_{0,\sqrt{a}}$. ✓

<u>Step 2</u>: Show $\mu$ is surjective.

We know: $\quad$ * $\mathrm{Isom}(\mathbb{H}^2)$ is <u>generated</u> by hyperbolic reflections

* $\mu$ is a homomorphism

This means that it suffices to show that every hyperbolic reflection is a Möbius transformation.

Hyperbolic reflections come in two types.

1. Across vertical lines: $\quad z \longmapsto -\bar{z} + a$
2. Circle inversions: $\quad I_{A,k} = \tau_A \circ I_{0,k} \circ \tau_A^{-1} : z \longmapsto \frac{k^2}{\bar{z} - a} + a$,

   where $A = (a, 0)$.

Both of these are Möbius transformations. $\square$

<u>Theorem</u> (Hsu 6.2.4): $\ker(\mu) = Z(GL_2(\mathbb{R}))$.

<u>Proof</u>: Exercise (HW 7).

<u>Cor</u>: $\mathrm{Isom}(\mathbb{H}^2) \cong PGL_2(\mathbb{R})$.

<u>Proof</u>: This is simply an application of the $1^{st}$ isomorphism theorem: If $\varphi: G \to H$ is a homomorphism, then $G/\ker\varphi \cong \mathrm{im}\,\varphi$.

We have $\mu: GL_2(\mathbb{R}) \longrightarrow \mathrm{Isom}(\mathbb{H}^2)$

and $\ker\mu = Z(GL_2(\mathbb{R}))$ (Thm 6.2.4)

$\mathrm{im}\,\mu = \mathrm{Isom}(\mathbb{H}^2)$ (Thm 6.2.3).

By the $1^{st}$ isom. thm:

$$\mathrm{Isom}(\mathbb{H}^2) \cong GL_2(\mathbb{R}) \big/ Z(GL_2(\mathbb{R})) \cong PGL_2(\mathbb{R}).$$

$\square$

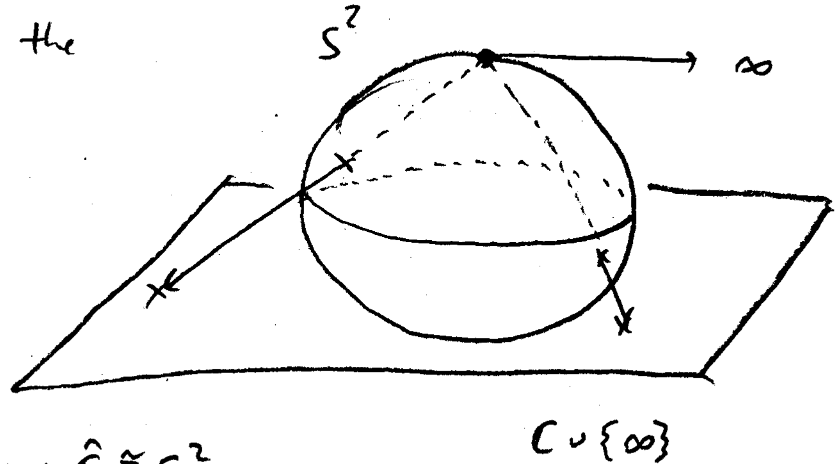
<u>Remark</u>: The bijection $\Phi: \mathcal{Z}/\sim \longrightarrow \hat{\mathbb{C}}$ has the interesting interpretation:

Think of $\mathcal{Z}/\sim$ as the set of <u>lines through $\vec{0}$</u> in $\mathbb{C}^2$.

(i.e., all one-dimensional complex subspaces in the 2-dimensional vector space $\mathbb{C}^2$.)

In this context, we call $\mathcal{Z}/\sim$ the <u>complex projective line</u>, denoted $\mathbb{CP}^1$.

As we've seen, the set $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is topologically the same (homeomorphic) to the 2-sphere, $S^2$.




Recall "Stereographic projection":

The map $\Phi$ extends to a bijection $Z/_\sim = \mathbb{CP}^1 \longrightarrow \hat{\mathbb{C}} \cong S^2$.

In fact, the map $\Phi : \mathbb{CP}^1 \to S^2$ and its inverse are continuous, meaning that $\Phi$ is a homeomorphism between $\mathbb{CP}^1$ and $S^2$.