8. Möbius transformations & the hyperbolic isometry group

Goal: Classify hyperbolic isometries both analytically and algebraically (i.e., as quotients of matrix groups.)

We've already seen some hyperbolic isometries:

* Horizontal translation: \( f(z) = z + \ell \)

* Reflection in vertical line: \( f(z) = -\overline{z} + \ell \)

* Inversion \( I_0,\ell \) : \( f(z) = \frac{k^2}{\overline{z}} \)

* Inversion \( I_C,\ell : \Gamma_C \circ I_{0,\ell} \circ \Gamma_C \) : \( f(z) = \frac{k^2}{\overline{z} - C} + C \)

Composing two reflections

Examples:

(i) \( I_0,2 : z \to \frac{z}{2} \)

\[
1+i \to \frac{4}{1+i} = \frac{4}{1-i} \cdot \frac{1+i}{1+i} = \frac{4+4i}{2} = 2+2i
\]

(ii) Let \( A = (3,0) \). \( I_A,4 : z \to \frac{4}{z-3} + 3 = \frac{16}{z-3} + \frac{3(z-3)}{z-3} = \frac{3z+7}{z-3} \)

\[
1+i \to \frac{3(1+i)+7}{1+i-3} = \frac{10-3i}{-2-i} \cdot \frac{-2+i}{-2+i} = \frac{-17+16i}{5}
\]

Note: The axes of these two reflections intersect.
The composition \( R := \Gamma_{A,H} \circ \Gamma_{0,2} \)

is a "hyperbolic rotation"

(see the "absolute proposition")

Prop 2.2.6 in Stahl: \( \rho_m \circ \rho_n = R_{A,2\alpha} \), where \( \alpha = \frac{\pi}{4}(n,m) \).

\[ R(z) = \Gamma_{A,H} \circ \Gamma_{0,2} (z) = \Gamma_{A,H} \left( \frac{y}{z} \right) = \frac{\frac{4}{2} + 7}{\frac{4}{2} - 3} = \frac{12 + 12}{-3 + 4} \]

The fixed point is found by solving \( \frac{7z + 12}{-3z + 4} = z \)

\[ 7z + 12 = -3z^2 + 4z \]

\[ 7z + 12 = z \]

\[ \Rightarrow 2z + 4 = 0 \]

\[ \Rightarrow z = -\frac{2}{2} = -1^2 \frac{\sqrt{15}}{2} \]

Only \( -\frac{1 + i\sqrt{15}}{2} \) lies in the upper half-plane.

The angle of rotation is \( 2\alpha \)

where \( \alpha = \cos^{-1} \left( \frac{OA^2 + OC^2 - 2\cdot AC \cdot CO}{2 \cdot AC \cdot CO} \right) \)

\[ = \cos^{-1} \left( \frac{4^2 + 2^2 - 3^2}{2 \cdot 4 \cdot 2} \right) = \cos^{-1} \left( \frac{11}{16} \right) \]

So \( \theta = 2\alpha \approx 93.15^\circ \).

(2) Consider \( T = \Gamma_{B,H} \circ \Gamma_{0,2} \), where \( B = (-1,0) \)

Since the axes of reflection don't intersect, this is in some sense a "hyperbolic translation."
Theorem (Stahl 9.2.4): The isometries of $H^2$ are precisely those functions $C \to C$ that can be written like

(i) $f(z) = \frac{\alpha z + \beta}{\delta z + \gamma}$, or
(ii) $f(z) = \frac{\alpha(-\overline{z}) + \beta}{\delta(-\overline{z}) + \gamma}$,

where $\alpha, \beta, \sigma, \delta \in \mathbb{R}$ satisfy $\alpha \delta - \beta \gamma > 0$.

We'll actually state and prove a stronger, more algebraic (and elegant!) version of this.

Examples of hyperbolic isometries in this form:

- Horizontal translations: $z \mapsto \frac{1 z + \gamma}{\delta z + 1}$
- Reflections across vertical geodesic: $z \mapsto \frac{1 (-z) + \gamma}{\delta(-z) + 1}$
- Reflections across bounded geodesic: $z \mapsto \frac{k^2}{\overline{z} - a} + a = \frac{q z + k^2 - a^2}{z - a} = \frac{-a(-\overline{z}) + (k^2 - a^2)}{z - a} = \frac{-1(-\overline{z}) - a}{a}$
Throughout, $F$ will be the field $\mathbb{R}$ or $\mathbb{C}$, $F^* := F \setminus \{0\}$, and $\hat{F} := F \cup \{\infty\}$.

**Def:** The **general linear group**, denoted $GL_n(\mathbb{R})$, is the set of $n \times n$ invertible matrices. It is a group under multiplication.

The **center** of $GL_n(\mathbb{R})$, denoted $Z(GL_n(\mathbb{R}))$, is the subgroup of elements that commute with everything. These are just the scalars of the identity matrix. That is:

$$Z(GL_n(\mathbb{R})) := \{ A \in GL_n(\mathbb{R}) : AB = BA \text{ for all } B \in GL_n(\mathbb{R}) \}$$

$$= \{ c I : c \in \mathbb{R}^* \}.$$  

It is well-known that $Z(GL_n(\mathbb{R})) \trianglelefteq GL_n(\mathbb{R})$, which means the the **quotient group** $GL_n(\mathbb{R})/Z(GL_n(\mathbb{R}))$ is a well-defined group. This group is called the **projective linear group**, and denoted $PGL_n(\mathbb{R})$.

It consists of the set of invertible $n \times n$ matrices under the equivalence that $A \sim B$ iff $B = cA$ for some $c \in \mathbb{R}^*$, i.e., the elements of $GL_n(\mathbb{R})$ up to scalar multiple.
Def: An **n x n** real **elementary matrix** is one of the following transformations on the column vectors of \( \mathbb{R}^n \); \( i \neq j \)

1. Exchanging the \( i \)-th and \( j \)-th coordinate:
   \[
   \begin{bmatrix}
   e_1 & \ldots & e_{i-1} & e_i & e_{i+1} & \ldots & e_n
   \end{bmatrix}
   \]
   Here: \( \tau = (i, j) \in S_n \) \( (k \neq 0) \)

2. Adding \( a \times (\text{column } i) \) to column \( j \):
   \[
   \begin{bmatrix}
   e_1 & \ldots & e_{j-1} & (e_j + ae_i) & e_{j+1} & \ldots & e_n
   \end{bmatrix}
   \]

3. Multiplying column \( i \) by \( a \neq 0 \):
   \[
   \begin{bmatrix}
   e_1 & \ldots & e_{i-1} & ae_i & e_{i+1} & \ldots & e_n
   \end{bmatrix}
   \]

**Remarks:**

* Elementary matrices are invertible, as they correspond precisely to "elementary column operations" that put a matrix in reduced row echelon form (RREF).

* The inverse of an elementary matrix is an elem. matrix.

* Left-multiplication by elementary matrices correspond to elementary row operations.

**Example:** The \( 2 \times 2 \) elementary matrices are (for \( a \in \mathbb{R} \)):

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
a & 1
\end{pmatrix},
\begin{pmatrix}
a & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & a
\end{pmatrix}
\]

**Theorem (Hsu 6.1.3):** The non-real elementary matrices generate \( GL_n(\mathbb{R}) \).
Proof (sketch): Every invertible $n \times n$ matrix may be row-reduced to the identity matrix.

Algebraically, there is a sequence $E_1, \ldots, E_k$ of elementary matrices such that $E_k \ldots E_2 E_1 A = I$

$\Rightarrow A = E_1^{-1} E_2^{-1} \ldots E_k^{-1}$.

Remark: \[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
\]

and \[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

Cor (Hsu 6.1.4): $\text{GL}_n(\mathbb{R})$ is generated by the matrices \[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix},
\text{ and } \begin{pmatrix}
a & 0 \\
0 & a \\
\end{pmatrix}
\text{ for all } a \neq 0.
\]

Recall: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ and $\text{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$

Also, $\det(A - \lambda I) = \lambda^2 - (\text{tr} A) \lambda + \det A$

$\det(AB) = (\det A)(\det B)$.

Theorem (Hsu 6.1.5): $\det(A \text{P} \text{A}^{-1}) = \det A$, $\text{tr}(A \text{P} \text{A}^{-1}) = \text{tr} A$, i.e., determinant and trace are conjugacy class invariants in $\text{GL}_2(\mathbb{R})$. 

Remark: If $A \in GL_2(\mathbb{R})$, then $\det(A) = k^{2} \det(A)$, $\text{tr}(kA) = k \text{tr}(A)$. Therefore, $\det$ and $\text{tr}$ are not well-defined notions in $PGL_2(\mathbb{R})$, i.e., equivalent elements may have different values.

We can fix this: define $f(A) = \frac{(\text{tr}(A))^2}{\det A}$.

Note that $f(kA) = f(A)$, so $f$ is well-defined on $PGL_2(\mathbb{R})$.

By Theorem 6.1.5, we have:

Cor 6.1.6: $f$ is a conjugacy class invariant of $PGL_2(\mathbb{R})$.

**Def:** The Möbius map $\mu$ is the function that sends $A \in GL_2(\mathbb{R})$ to the function $\mu_A$, where

$$
\mu_A(z) = \begin{cases} 
\frac{az + b}{cz + d} & \text{det } A > 0 \\
\frac{\bar{a} \bar{z} + \bar{b}}{\bar{c} \bar{z} + \bar{d}} & \text{det } A < 0
\end{cases}
$$

**Concern:** The function isn't defined for all of $\mathbb{C}$, because the denominator could be zero.

We can fix this too.
Let \( \mathbb{Z} = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} : z_i \in \mathbb{C} \right\} \)

Consider the "bar" map: \( \mathbb{Z} \to \mathbb{Z} \),
\[
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}.
\]

Observation: \( x \begin{pmatrix} \overline{z} \\ z \end{pmatrix} = \begin{pmatrix} \overline{z} \\ z \end{pmatrix} \), i.e., \( x^{-1} = id \).

\( y \begin{pmatrix} \overline{z} \\ z \end{pmatrix} = A \overline{z} \) if \( A \in GL_2(\mathbb{R}) \).

Define an action \( \text{of } GL_2(\mathbb{R}) \) on \( \mathbb{Z} \) by the rule:
\[
A \cdot z = \begin{cases} 
A \overline{z} & \text{if } \det A > 0 \\
A \overline{z} & \text{if } \det A < 0.
\end{cases}
\]

We must check that \((AB) \cdot z = A \cdot (B \cdot z)\).

There are 4 cases:

1. \( \det A, \det B > 0 \) : \((AB) \cdot z = AB \overline{z} = A \cdot (B \cdot z) \) \( \checkmark \)
2. \( \det A > 0, \det B < 0 \) : \((AB) \cdot z = AB \overline{z} = A \cdot (B \cdot z) \) \( \checkmark \)
3. \( \det A < 0, \det B > 0 \) : \((AB) \cdot z = AB \overline{z} = A \overline{B} \overline{z} = A \cdot (B \cdot z) \) \( \checkmark \)
4. \( \det A, \det B < 0 \) : \((AB) \cdot z = AB \overline{z} = A \overline{B} \overline{z} = A \overline{B} \overline{z} = A \cdot (B \cdot z) \) \( \checkmark \)
Remark: The action $A_z$ preserves the "non-zero scalar multiple" equivalence on $\mathbb{Z}$:

Take $A \in \text{GL}_2(\mathbb{R})$, $\lambda \in \mathbb{C}^*$, $z \in \mathbb{Z}$.

$A\lambda z = \lambda Az$ and $\overline{\lambda z} = \overline{\lambda \overline{z}}$, and so

$A\cdot (\lambda z) = \lambda' (A \cdot z)$ for some $\lambda' \in \mathbb{C}$.

Thus, this action carries over to an action on $\mathbb{Z}/\mathbb{N}$.

Key idea: There is a natural bijection

$\Phi : \mathbb{Z}/\mathbb{N} \longrightarrow \mathbb{C}$, $\lambda : \lambda(\overline{z}) \longmapsto \begin{cases} \frac{z_1}{z_2} & z_2 \neq 0 \\ \infty & z_2 = 0 \end{cases}$

This is well-defined because $\frac{z_1}{z_2} = \frac{\lambda z_1}{\lambda z_2}$.

We still need to show (4.w).

1. $\Phi$ is a bijection
2. Given the conventions that $\frac{a \infty + b}{c \infty + d} = \frac{a}{c}$ and $\frac{a}{0} = \infty$,

then for $z \in \mathbb{Z}/\mathbb{N}$, we have $\mu_A(\overline{z}) = \overline{\lambda (A \cdot z)}$.

Another way to see this:

Recall that an action of a group $G$ on a set $X$ is a homomorphism $\phi : G \rightarrow S_X$.

\[
\begin{array}{ccc}
\mathbb{Z}/\mathbb{N} & \xrightarrow{\text{action by } A} & \mathbb{Z}/\mathbb{N} \\
\Phi & \downarrow & \Phi \\
\hat{G} & \xrightarrow{\text{action by } A} & \hat{G}
\end{array}
\]
Theorem (Hsu 6.2.2): Given the conventions that \( \frac{a \infty + b}{c \infty + d} = \frac{a}{c} \)
and \( \frac{a}{0} = \infty \), the Möbius map is a homomorphism
\[ \mu : \text{GL}_2(\mathbb{R}) \to S_\infty \] 
(set of permutations of \( \infty \)). In other words, \( \mu \) defines an action of \( \text{GL}_2(\mathbb{R}) \) on \( \infty \).

Proof: To show that \( \mu \) is indeed a homomorphism, we must show that for any \( z \in \mathbb{Z}/\mathbb{Z} \), \( \mu_{AB}(\overline{\Phi}(z)) = \mu_A \mu_B(\overline{\Phi}(z)) \). This holds because
\[
\mu_{AB}(\overline{\Phi}(z)) = \overline{\Phi}(AB, z) = \overline{\Phi}(A, B, z) = \mu_A \overline{\Phi}(B, z) = \mu_A \mu_B(\overline{\Phi}(z)). \]

Theorem (Hsu 6.2.3): The Möbius map is a surjective (onto) homomorphism \( \mu : \text{GL}_2(\mathbb{R}) \to \text{Isom}(\mathbb{H}^2) \leq S_\infty \).

Proof: We need to show 2 things:

1. Show that for any \( A \in \text{GL}_2(\mathbb{R}) \), \( \mu A \in \text{Isom}(\mathbb{H}^2) \).

2. Show that \( \mu \) maps onto \( \text{Isom}(\mathbb{H}^2) \).

Step 1: Since \( \mu \) is a homomorphism, it suffices to prove (1) just for the generators of \( \text{GL}_2(\mathbb{R}) \).

1. \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). \( \mu_A(z) = z + a \) is a horizontal translation.
(ii) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. \quad M_A(z) = \frac{1}{z} \quad \text{is the inversion $I_{0,1}$}

(iii) $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \; a > 0$. \quad A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}

\quad \Rightarrow M_A = I_{0,1} \circ I_{0,a}$

(iv) $A = \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}, \; a < 0$. \quad A = \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}

Note that $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the reflection in the vertical line $x=0$ and $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mapsto I_{0,1} \circ I_{0,a}$.

**Step 2:** Show $\mu$ is surjective.

We know: $\pi \text{ Isom}(\mathbb{H}^2)$ is generated by hyperbolic reflections.

$\pi \mu$ is a homomorphism.

This means that it suffices to show that every hyperbolic reflection is a Möbius transformation.

Hyperbolic reflections come in two types:

1. Across vertical lines: $z \mapsto -\overline{z} + a$

2. Circle inversions: $I_{A, z} = \tau_A \circ I_{0,a} \circ \tau_A^{-1}; z \mapsto \frac{k^2}{\overline{z} - a} + a$,

where $A = (a, 0)$.

Both of these are Möbius transformations.
Theorem (Hsu 6.2.4): \( \ker(\mu) = Z(GL_2(\mathbb{R})) \).

Proof: Exercise (Hw 7).

Cor: \( \text{Isom}(\mathbb{H}^2) \cong \text{PGL}_2(\mathbb{R}) \).

Proof: This is simply an application of the first isomorphism theorem: If \( \phi: G \rightarrow H \) is a homomorphism, then \( G/\ker(\phi) \cong \text{im}(\phi) \).

We have \( \mu: GL_2(\mathbb{R}) \rightarrow \text{Isom}(\mathbb{H}^2) \)

and \( \ker(\mu) = Z(GL_2(\mathbb{R})) \) \((\text{Thm 6.2.4})\)

in \( \mu = \text{Isom}(\mathbb{H}^2) \) \((\text{Thm 6.2.3})\).

By the 1st isom. thm:

\[
\text{Isom}(\mathbb{H}^2) \cong GL_2(\mathbb{R})/Z(GL_2(\mathbb{R})) \cong \text{PGL}_2(\mathbb{R}).
\]

Remark: The bijection \( \mathbb{C}/\mathbb{R} \rightarrow \mathbb{C} \) has the interesting interpretation:

Think of \( \mathbb{C}/\mathbb{R} \) as the set of lines through \( 0 \) in \( \mathbb{C}^2 \)

(i.e., all one-dimensional complex subspaces in the 2-dimensional vector space \( \mathbb{C}^2 \))

In this context, we call \( \mathbb{C}/\mathbb{R} \) the complex projective line,

denoted \( \mathbb{CP}^1 \).
As we've seen, the set $\hat{C} = CV\{\infty\}$ is topologically the same (homeomorphic) to the 2-sphere, $S^2$.

Recall "Stereographic projection":

The map $\Phi$ extends to a bijection $\mathbb{C}/\mathbb{Z} \cong CP^1 \rightarrow \hat{C} \cong S^2$.

In fact, the map $\Phi: CP^1 \rightarrow S^2$ and its inverse are continuous, meaning that $\Phi$ is a homeomorphism between $CP^1$ and $S^2$. 