14. Curvature and the Gauss-Bonnet theorem

Throughout, we'll say that a triangulated surface to be a surface homeomorphic to a triangular complex (not necessarily simplicial).

Def: A triangulated surface S is a geometrized polyhedron if each triangle is assigned a number at each vertex (think: angle).

For any vertex $v \in S$, define the curvature at v to be

$$\kappa(v) = 2\pi - \sum_{\text{angles } A @ v} A \quad , \quad \text{i.e., the "angle deficit"}$$

For any triangle $f \in S$, define the curvature over f to be

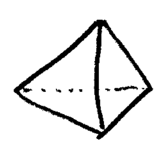
$$\int_f \kappa = A + B + C - \pi, \quad \text{where } A, B, C \text{ are the angles of } f.$$

Define the curvature of a geometrized polyhedron S to be

$$\int_S \kappa = \sum_{f \in S} \int_f \kappa + \sum_{v \in S} \kappa(v)$$

Examples:

(1) Let S be a regular tetrahedron:

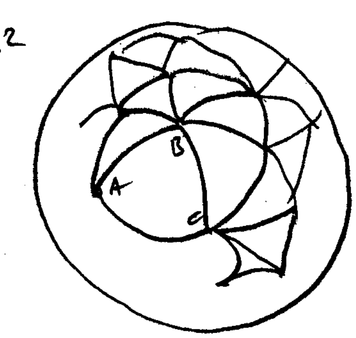


For any $v \in S$: $\kappa(v) = 2\pi - 3 \cdot \frac{\pi}{3} = \pi$

For any $f \in S$: $\int_f \kappa = 0$ (faces are "flat")

2

S^2



(ii) Let S be a triangulated sphere:

For any vertex v: $\kappa(v) = 0$

"In general," $\int_f \kappa > 0$ (Spherical triangles have angles that sum to more than 2π).

<u>Theorem</u> (Gauss-Bonnet): If S is a geometrized polyhedron then $\int_S \kappa = 2\pi \, \chi(S)$.

<u>Proof</u>: Let V, E, F be the number of vertices, edges, & faces of S.

For any face $f \in S$, let A_f, B_f, C_f be its angles.

By definition: $\int_S \kappa = \sum_{f \in S} \int_f \kappa + \sum_{v \in S} \kappa(v)$

$$= \sum_{f \in S} (A_f + B_f + C_f - \pi) + \sum_{v \in S} \kappa(v)$$

$$= -\pi F + \left(\sum_{\text{angles } A \in S} A \right) + \sum_{v \in S} \kappa(v). \qquad (*)$$

<u>Substitute</u>:
- $\sum_{A \in S} A = \sum_{v \in S} (2\pi - \kappa(v)) = 2\pi V - \sum_{v \in S} \kappa(v)$
- $3F = 2E$ (Every face touches 3 edges, but this double-counts each edge)

$\Rightarrow F = 2E - 2F$

Plug these back into (*), yields

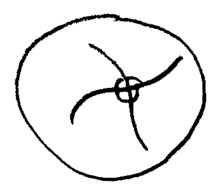
$$\int_S \kappa = -\pi F + \left(\sum_{A \in S} A\right) + \sum_{v \in S} \kappa(v)$$

$$= -\pi(2E - 2F) + \left(2\pi V - \sum_{v \in S} \kappa(v)\right) + \sum_{v \in S} \kappa(v)$$

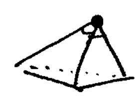
$$= 2\pi(V - E + F) = 2\pi\,\chi(S). \qquad \square$$

Remarks:

* If S is a smooth surface, then $\kappa(v) = 2\pi$, but $\int_f \kappa$ is typically not 2π



* If S is a polyhedron, then $\kappa(v)$ is typically not 2π but $\int_f \kappa = 0$ (because facets are flat).

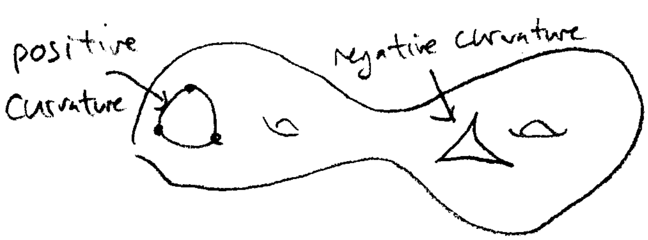


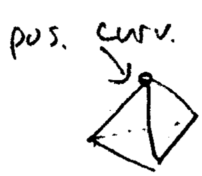
Thus, a corollary of the Gauss-Bonnet theorem is:

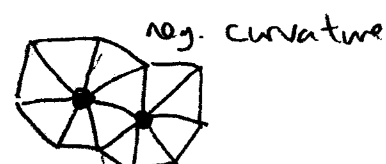
Cor: If S is a triangulated smooth surface, then

$$\int_S \kappa = \sum_{f \in S} \int_f \kappa = 2\pi\,\chi(S).$$

If S is a polytope, then $\int_S \kappa = \sum_{v \in S} \kappa(v) = 2\pi\,\chi(S)$.



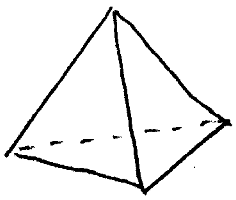




(7 equilateral Δ^s at a vertex)

Examples: Tetrahedron

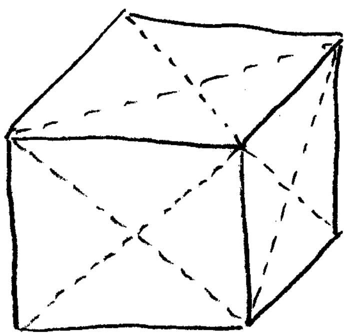
$$\int_S \kappa = \sum_{f \in S} \left(\frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} - \pi\right) + \sum_{v \in S} 2\pi - 3 \cdot \frac{\pi}{3}$$



$\simeq S^2$

$$= 4(0) + 4(\pi) = 4\pi = 2\pi\chi(S^2) \checkmark$$

Cube: First, we need to triangulate to make it a triangle complex.

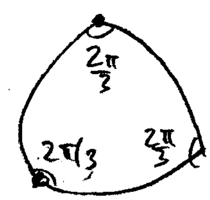


Clearly, $\int_f \kappa = 0$ for each face.

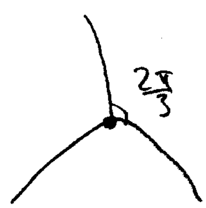
- 8 "corner" vertices with $\kappa(v) = 2\pi - 3 \cdot \frac{\pi}{2} = \frac{\pi}{2}$
- 6 "center" vertices with $\kappa(v) = 0$.

Together, $\int_S \kappa = 24(0) + 6(0) + 8\left(\frac{\pi}{2}\right) = 4\pi = 2\pi\chi(S^2)$. ✓

"Geometrized" tetrahedron: Build with 4 copies of



Each vertex:

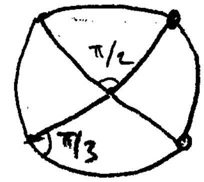


Clearly, $\kappa(v) = 0$.

For each face: $\int_f \kappa = 3 \cdot \frac{2\pi}{3} - \pi = \pi$.

Together, $\int_S \kappa = \sum_{f \in S} \kappa = 4\pi = 2\pi\chi(S^2)$ ✓

"Geometrized" cube: Build with 6 copies of



For each vertex: $K(v) = 0$

For each face: $\int_F K = \frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{3} - \pi = \frac{\pi}{6}$ $(F = 24)$.

Together: $\int_S K = \sum_{F \in S} \frac{\pi}{6} = 24 \cdot \frac{\pi}{6} = 4\pi = 2\pi\,\chi(S^2)$. —

Def: A geometrized triangle with angles A, B, C is hyperbolic if $A+B+C < \pi$, Euclidean if $A+B+C = \pi$, and spherical if $A+B+C > \pi$.

Def: The scalar curvature at any point on a geometrized triangle is -1 if it's hyperbolic, 0 if it's Euclidean, and $+1$ if it's spherical.

Recall that for a hyperbolic triangle T, $ha(T) = \pi - (A+B+C) = -\int_T K$.

* Fact: For any triangle T on the unit sphere with angles A, B, C,

spherical area$(T) = A+B+C-\pi = \int_T K$.

Theorem (Local Gauss-Bonnet) Let T be a triangle, and K the scalar curvature function. Then

$$\int_T K\,dA = A+B+C-\pi,$$

$$\text{where } dA = \begin{cases} dx\,dy & \text{if } T \text{ is Euclidean} \\ \dfrac{dx\,dy}{y} & \text{if } T \text{ is hyperbolic} \\ \text{surface area of unit sphere} & \text{if } T \text{ is spherical} \end{cases}$$

Proof:

Hyperbolic: $\int_T K\,dA = -\int_T dA = -ha(T) = A+B+C-\pi$

Euclidean: $\int_T K\,dA = \int_T 0\,dA = 0 = A+B+C-\pi$

Spherical: $\int_T K\,dA = \int_T dA = \text{spherical area}(T) = \int_T \kappa = A+B+C-\pi$

(see also Chapter 11 in Stahl for more on Spherical geometry.) $\square$

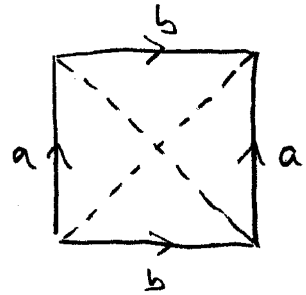
Def: Let S be a geometrized polyhedron. Call the geometric structure a smooth triangulation if, for all vertices $v \in S$, $\kappa(v) = 0$. A smooth triangulation has a hyperbolic (resp. Euclidean, or spherical) structure if:

1. All triangles of S are hyperbolic (resp. Euclidean or spherical).
2. There exist actual hyperbolic (resp. Euclidean, spherical) triangles with the specified angles such that the lengths of the edges of triangles that meet at an edge in S agree.

<u>Theorem</u> (Hsu 11.2.6): Let S be an orientable compact connected surface of genus g. If $g=0$, S has a <u>spherical</u> structure. If $g=1$, S has a <u>Euclidean</u> structure, and if $g \geq 2$, S has a <u>hyperbolic</u> structure.

Proof: $\boxed{g=0}$ Follows from our previous example of a geometrized tetrahedron.

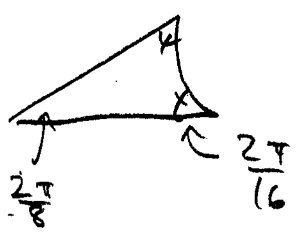
$\boxed{g=1}$ Consider a torus:

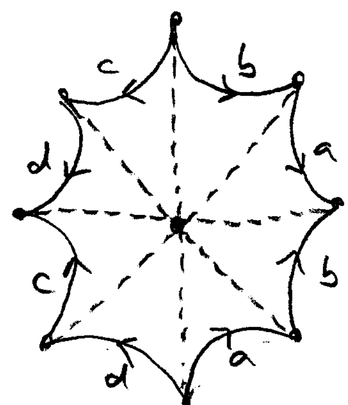


4 <u>Euclidean</u> triangles.

$\boxed{g=2}$ Consider a genus-2 torus, $T^2 \# T^2$:

Each triangle:





For any triangle, $A+B+C = \frac{2\pi}{8} + \frac{2\pi}{16} + \frac{2\pi}{16} = \frac{\pi}{2} < \pi$.

Note that $V=1$ and $E=4$ in this triangulation.

$\boxed{g>2}$ Similar picture as above, but now, $V=1$, $E=2g$, and each triangle has angles $\frac{2\pi}{4g}$, $\frac{2\pi}{8g}$, $\frac{2\pi}{8g}$, which sum to $\frac{\pi}{g} < \pi$. Thus the structure is <u>hyperbolic</u>. $\square$