14. Curvature and the Gauss-Bonnet theorem

Throughout, we'll say that a \textit{triangulated surface} to be a surface homeomorphic to a triangular complex (not necessarily simplicial).

**Def:** A triangulated surface $S$ is a \textit{generalized polyhedron} if each triangle is assigned a number at each vertex (think: angle).

For any vertex $v \in S$, define the \textit{curvature} at $v$ to be

$$K(v) = 2\pi - \sum_{\text{angle } A \in v} A,$$

i.e., the "angle deficit".

For any triangle $f \in S$, define the \textit{curvature} over $f$ to be

$$\int_f K = A + B + C - \pi,$$ where $A, B, C$ are the angles of $f$.

Define the \textit{curvature of a generalized polyhedron} $S$ to be

$$\int_S K = \sum_{f \in S} \int_f K + \sum_{v \in S} K(v).$$

**Examples.**

(1) Let $S$ be a regular tetrahedron:

For any $v \in S$: $K(v) = 2\pi - 3 \cdot \frac{\pi}{3} = \pi$

For any $f \in S$: $\int_f K = 0$ (faces are "flat").
Let $S$ be a triangulated sphere.

For any vertex $v$: $\chi(v) = 0$

"In general," $\int f \chi > 0$ (spherical triangles have angles that sum to more than $2\pi$).

Theorem (Gauss–Bonnet): If $S$ is a geometrized polyhedron

then $\int_S \chi = 2\pi \chi(S)$.

Proof: Let $V, E, F$ be the number of vertices, edges, and faces of $S$.

For any face $f \subseteq S$, let $A_f, B_f, C_f$ be its angles.

By definition: $\int_S \chi = \sum_{f \subseteq S} \int_f \chi + \sum_{v \subseteq S} \chi(v)$

$= \sum_{f \subseteq S} \left( A_f + B_f + C_f - \pi \right) + \sum_{v \subseteq S} \chi(v)$

$= -\pi F + \left( \sum_{\text{angle } a \subseteq S} A \right) + \sum_{v \subseteq S} \chi(v)$.

Substitute: $\sum_{\text{angle } a \subseteq S} A = \sum_{v \subseteq S} \left( 2\pi - \chi(v) \right) = 2\pi V - \sum_{v \subseteq S} \chi(v)$

* $3F = 2E$ (Every face touches 3 edges, but this double-counts each edge)

$\Rightarrow F = 2E - 2F$
Plug these back into (4), yields

\[
\int_S \chi = -\pi F + \left( \sum_{A \in S} A \right) + \sum_{V \in S} \chi(V)
\]

\[
= -\pi (2E-2F) + \left( 2\pi V - \sum_{V \in S} \chi(V) \right) + \sum_{V \in S} \chi(V)
\]

\[
= 2\pi (V-E+F) = 2\pi \chi(S).
\]

Remarks:

+ If \( S \) is a smooth surface, then \( \chi(V) = 2\pi \), but \( \int_f \chi \) is typically not \( 2\pi \).

* If \( S \) is a polyhedron, then \( \chi(V) \) is typically not \( 2\pi \)

but \( \int_f \chi = 0 \) (because facets are flat).

Thus, a corollary of the Gauss-Bonnet theorem is:

Cor: If \( S \) is a triangulated smooth surface, then

\[
\int_S \chi = \sum_{f \in S} \int_f \chi = 2\pi \chi(S).
\]

If \( S \) is a polytope, then \( \int_S \chi = \sum_{V \in S} \chi(V) = 2\pi \chi(S) \).
Examples: Tetrahedron

\[ \int_s \chi = \sum_{f \in S} \left( \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} - \pi \right) + \sum_{v \in S} 2\pi - 3 \cdot \frac{\pi}{3} \]

\[ = 4(0) + 4(\pi) = 4\pi = 2\pi \chi(S^2) \]

Cube

First, we need to triangulate to make it a triangle complex.

Clearly, \( \int_f \chi = 0 \) for each face.

- 8 "corner" vertices with \( \chi(v) = 2\pi - 3 \cdot \frac{\pi}{2} = \frac{\pi}{2} \)
- 6 "center" vertices with \( \chi(v) = 0 \).

Together, \( \int_s \chi = 24(0) + 6(0) + 8 \left( \frac{\pi}{2} \right) = 4\pi = 2\pi \chi(S^2) \).

"Geometricized" tetrahedron:  
Build with 4 copies of

Each vertex: \( \frac{2\pi}{3} \)

Clearly, \( \chi(v) = 0 \).

For each face: \( \int_f \chi = 3 \cdot \frac{2\pi}{3} - \pi = \frac{\pi}{3} \).

Together, \( \int_s \chi = \sum_{f \in S} \chi = 4\pi = 2\pi \chi(S^2) \).
"Geometrized cube": Build with 6 copies of $\frac{\pi}{2 \times \frac{\pi}{3} + \frac{\pi}{3} - \pi}{\frac{\pi}{6}} = 2\pi \chi(S^2).

For each vertex: $\chi(v) = 0$

For each face: $\int_f \chi = \frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{3} - \pi = \frac{\pi}{6} \quad (F = 24)$.

Together: $\int_S \chi = \sum_{f \in F} \frac{\pi}{6} = 24 \cdot \frac{\pi}{6} = 4\pi = 2\pi \chi(S^2)$.

**Def:** A geometrized triangle with angles $A, B, C$ is **hyperbolic** if $A + B + C < \pi$, **Euclidean** if $A + B + C = \pi$, and **spherical** if $A + B + C > \pi$.

**Def:** The **scalar curvature** at any point on a geometrized triangle is $-1$ if it's hyperbolic, $0$ if it's Euclidean, and $+1$ if it's spherical.

Recall that for a hyperbolic triangle $T$, $\chi(T) = \pi - (A + B + C) = -\int_T \chi$.

**Fact:** For any triangle $T$ on the unit sphere with angles $A, B, C$,

$spherical \ a rea(T) = A + B + C - \pi = \int_T \chi$.

**Theorem (Local Gauss–Bonnet):** Let $T$ be a triangle, and $K$ the scalar curvature function. Then

$\int_T K \ dA = A + B + C - \pi$.
where \( dA = \begin{cases} \frac{dx\,dy}{y} & \text{if } T \text{ is hyperbolic} \\ \text{surface area of unit sphere} & \text{if } T \text{ is spherical} \end{cases} \)

**Proof:**

- **Hyperbolic:** \( \int_T K\,dA = -\int_T \text{d}A = -\text{ha}(T) = A + B + C - \pi \)

- **Euclidean:** \( \int_T K\,dA = \int_T 0\,dA = 0 = A + B + C - \pi \)

- **Spherical:** \( \int_T K\,dA = \int_T \text{d}A = \text{spherical area}(T) = \int_T \chi = A + B + C - \pi \)

(see also Chapter 11 in Stahl for more on Spherical geometry.)

**Def:** Let \( S \) be a geometrized polyhedron. Call the geometric

structure a **smooth triangulation** if, for all vertices \( v \in S \),

\( X(v) = 0 \). A smooth triangulation has a hyperbolic (resp. Euclidean, or spherical) structure if:

1. All triangles of \( S \) are hyperbolic (resp. Euclidean or spherical).
2. There exist actual hyperbolic (resp. Euclidean, spherical) triangles with the specified angles such that the lengths of the edges of triangles that meet at an edge in \( S \) agree.
Theorem (Hsu II.2.6): Let $S$ be an orientable compact connected surface of genus $g$. If $g=0$, $S$ has a \textit{spherical} structure. If $g=1$, $S$ has a \textit{Euclidean} structure, and if $g \geq 2$, $S$ has a \textit{hyperbolic} structure.

Proof. ($g=0$) Follows from our previous example of a geometrically tetrahedron.

$g=1$\quad Consider a torus:

4 Euclidean triangles.

$g=2$\quad Consider a genus-2 torus, $T^2 \# T^2$:

Each triangle:

\[ \frac{2\pi}{8} < \text{angle} < \frac{4\pi}{8} \]

For any triangle, $A+B+C = \frac{2\pi}{8} + \frac{2\pi}{8} + \frac{2\pi}{8} = \frac{\pi}{2} < \pi$.

Note that $V=1$ and $E=4$ in this triangulation.

$g \geq 2$\quad Similar picture as above, but now, $V=1$, $E=2g$, and each triangle has angles $\frac{2\pi}{4g}$, $\frac{2\pi}{8g}$, $\frac{2\pi}{8g}$, which sum to $\frac{\pi}{g} < \pi$. Thus the structure is \textit{hyperbolic}. \qed