1. In our first model of air resistance, the resistance force depends only on the velocity. This leads to the differential equation $mv' = -mg - rv$, where $-mg$ is the gravitation force and $R(v) = -rv$ is the air resistance force. For an object that drops a considerable distance, such as a parachutist, there is a dependence on the altitude as well. It is reasonable to assume that the resistance force is proportional to air pressure, as well as velocity. Furthermore, to a first-order approximation, the air pressure varies exponentially with the altitude (i.e., it is proportional to e^{-ax}, where a is a constant and x is the altitude). Propose and justify (but do not solve) a differential equation model (in x instead of v) for the velocity of a falling object subject to such a resistance force. Recall that $x' = v$.

2. The population of snakes on a plane is believed to be growing according to the logistic equation:
$$y' = ry\,(1 - y/M) \qquad\qquad \text{Solution } y(t) = \frac{M}{1 + Ce^{-rt}}\,.$$
The maximum number of snakes that can live on the plane is 1000. Initially, the population is 500, and at this time, the rate of increase of snakes is 100 per month.
   (a) How many months until the population reaches 90% of the maximum?
   (b) Sketch this solution curve in the (t, y)-plane, and the two steady-state solutions.

3. Let $T(t)$ be the temperature of a cup of water at time t, in hours. Newton's law of cooling says that at any t, the rate of change $T'(t)$ is proportional to the difference in ambient temperature and $T(t)$, which can be described by the differential equation
$$T' = k(A - T).$$
The ambient temperature A need not be constant; suppose it varies sinusoidally with time with a period of 24 hours. At 6am, the ambient temperature is at its minimum of 40° and at 6pm, its maximum of 60°.
   (a) Write down a differential equation (that is, find $A(t)$) that models the temperature of the cup of water. Let $t = 0$ be noon.
   (b) Give a physical *and* mathematical explanation why there is no steady-state solution.
   (c) The long-term behavior of a solution to this equation is a sinusoid; explain why. What do you expect the period, amplitude, and phase shift of this solution to be (qualitively) compared to $A(t)$?
   (d) Sketch a graph of several solutions to this equation corresponding to different initial temperatures.
   (e) Suppose that instead of measuring the temperature of a cup of water, $T(t)$ measures the temperature of a pond. What parameters would this change in the differential equation? What would your graphs in Part (d) look like? Sketch solutions corresponding to the same initial conditions.

4. A population is originally 100 individuals, but because of the combined effects of births and deaths, it triples each hour.
   (a) Make a table of population size of $t = 0$ to 5 where t is measured in hours.
   (b) Write a difference equation modeling the population growth two different ways: first by expressing P_{t+1} in terms of P_t, and then expressing ΔP in terms of P_t.
   (c) What, if anything, can you say about the birth and death rates for this population?

5. In the early stages of the development of a frog embryo, cell division occurs at a fairly regular rate. Suppose you observe that all cells divide, and hence the number of cells doubles, roughly every half-hour.
   (a) Write down a difference equation modeling this situation. You should specify how much real-world time is represented by an increment of 1 in t. Suppose the initial number of cells is 1.
   (b) Produce a table and graph of the number of cells as a function of t.
   (c) Further observation shows that after 10 hours, the embryo has around 30,000 cells. Is this roughly consistent with your model? What biological conclusions and/or questions does this raise?

6. Suppose the size of a certain population is affected only by birth, death, immigration, and emigration – each of which occurs in a yearly amount proportional to the size of a population. That is, if the population is P, within a time period of 1 year, the number of births is bP, deaths is dP, immigrants is iP, and emigrants is eP for some b, d, i, and e. Show that the population can be modeled by $\Delta P = rP$ and give a formula for r.

7. Four of the many common ways of writing the discrete logistic growth equation are

$$\begin{aligned} \Delta P &= rP(1 - P/K), \qquad & \Delta P &= sP(K - P), \\ \Delta P &= tP - uP^2, \qquad & P_{t+1} &= vP_t - wP_t^2\,. \end{aligned}$$

   Write each of the following in all four of these forms.
   (a) $P_{t+1} = P_t + .2P_t(10 - P_t)$
   (b) $P_{t+1} = 2.5P_t - .2P_t^2$