2 Difference equations

Example: Each day, a population reproduces.

It has: "growth rate" of f

"death rate" of d

$$\Delta P = fP - dP = (f-d)P$$

let $P_t = P(t) =$ population at time t.

Then $\Delta P = P_{t+1} - P_t$

$$\Rightarrow P_{t+1} = P_t + \Delta P = P_t + (f-d)P_t = (1+f-d)P_t$$

or just $\boxed{P_{t+1} = \lambda P_t}$

Sol'n: $P_t = \lambda^t P_0$

Ex: $P_0 = 300$, $f = .03$, $d = .01$, $1+f-d = 1.02$

$P_1 = (1.02) P_0$

$P_2 = (1.02) P_1 = (1.02)^2 P_0$

$P_3 = (1.02) P_2 = (1.02)^3 P_0$

$\vdots$

2

* What is a difference eq'n?

Formally: Let Q be a quantity defined for all $t \in \mathbb{N}$ such that $Q_{t+1} = F(Q_t)$.

Previous ex: $F(x) = \lambda x$ "Malthusian model"

this is linear.

Goal: Find a good model

Analyze models that arise.

Difference eq'ns are "discrete time, continuous space"

(compare to differential eq'ns: "continuous time & space")

Logistic difference eq'n (one version)

Growth rate depends on size! (density dependent)

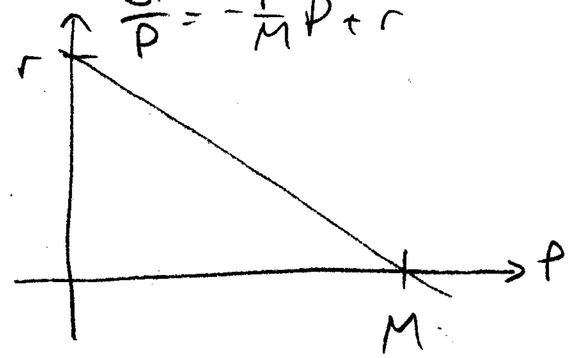
Death rate? Assume density independent. (why?)

Idea: Analyze $\frac{\Delta P}{P}$ = per capita growth rate.

P small: $\frac{\Delta P}{P}$ large

P large: $\frac{\Delta P}{P}$ small

P too large: $\frac{\Delta P}{P} < 0$.



$$\frac{\Delta P}{P} = -\frac{r}{M}P + r = r\left(1 - \frac{P}{M}\right).$$

or $P_{t+1} = P_t\left(1 + r\left(1 - \frac{P_t}{M}\right)\right)$ "discrete logistic model"

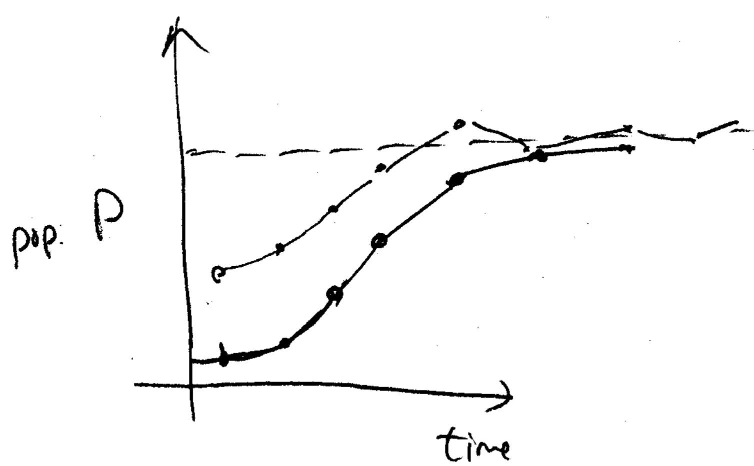
<u>Note</u>: • $P \ll K \Rightarrow 1 - \frac{P}{K} \approx 1 \Rightarrow P_{t+1} \approx (1+r)P_t$

• $P \approx K \Rightarrow P_{t+1} \approx 0$

<u>Question</u>: What is $F(x)$? (in class exercise)

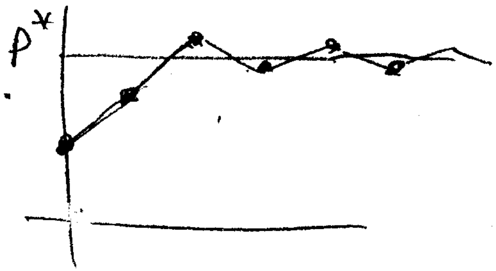
<u>Remark</u>: Though difference eqns are simple, they often have no closed form solution for P_t!

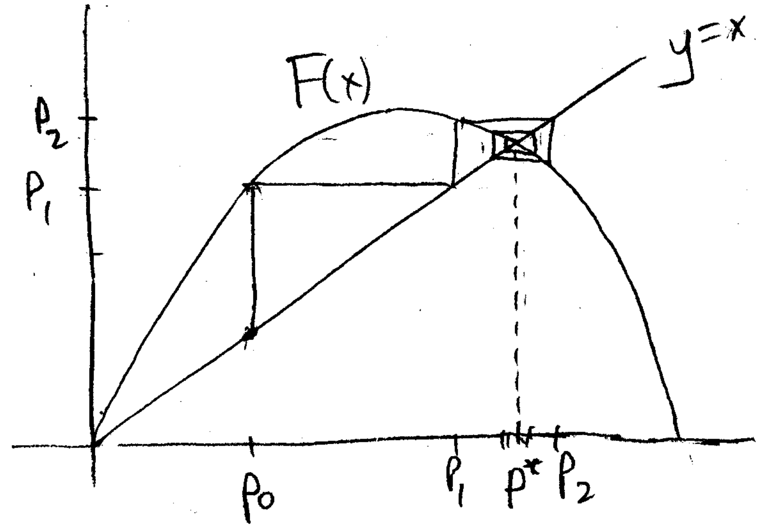
We can plot the solutions:



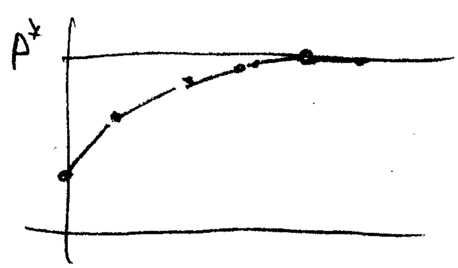
Cobwebbing: Suppose $P_{t+1} = F(P_t)$.

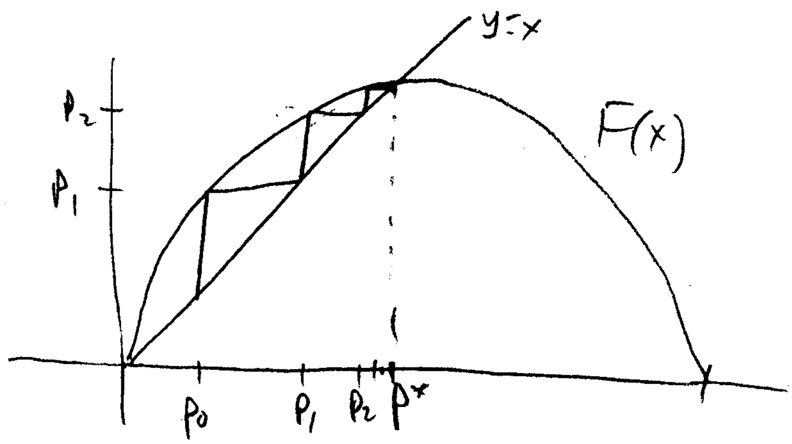
We can numerically find $P_0, P_1, P_2, \ldots$ as follows





VS.





Think: When will the first one (toggling) be more realistic?

" " " 2^{nd} " be more realistic?