Read: Lax, Chapter 1, pages 1–11.

1. (a) Show that there are no proper subfields of $\mathbb{Q}$.
   (b) Show that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a field.

2. Let X be a vector space over a field K. Let 0 be the zero element of K and $\mathbf{0}$ the zero-element of X. Using only the definitions of a group, a vector space, and a field, carefully prove each of the following:
   (a) The identity element e of a group is unique.
   (b) In any group G, the inverse of $g \in G$ is unique.
   (c) $0x = \mathbf{0}$ for every $x \in X$;
   (d) $k\mathbf{0} = \mathbf{0}$ for every $k \in K$;
   (e) For every $k \in K$ and $x \in X$, if $kx = \mathbf{0}$, then $k = 0$ or $x = \mathbf{0}$.

3. Let X denote the vector space of polynomials in $\mathbb{R}[x]$ of degree less than n. Are the vectors $x^3 + 2x + 5$, $3x^2 + 2$, $6x$, 6 linearly independent in X? (Assume that $n \geq 4$.)

4. The following is called the *Replacement Lemma*: Let X be a vector space over K, and let S be a linearly independent subset of X. Let $x_0 \in \operatorname{Span}(S)$ with $x_0 \neq 0$. Prove that there exists $x_1 \in S$ such that the set $S' = (S \setminus \{x_1\}) \cup \{x_0\}$ is a basis for $\operatorname{Span}(S)$.
   (a) Prove the Replacement Lemma
   (b) Suppose that B is a basis for X containing n elements, and let B' be another basis for X. Show that $|B'| = n$.

5. If Y is a subspace of X, then two vectors $x_1, x_2 \in X$ are *congruent modulo* Y, denoted $x_1 \equiv x_2 \bmod Y$, if $x_1 - x_2 \in Y$. This is an equivalence relation; denote the equivalence class containing $x \in X$ by $\{x\}$, and let X/Y denote the set of equivalence classes. We can make X/Y into a vector space by defining addition and scalar multiplication as follows:
$$\{x\} + \{z\} = \{x + z\}, \qquad \{ax\} = a\{x\}.$$
   Show that these operations are well-defined, that is, they do not depend on the choice of congruence class representatives.

6. Let S be a set of vectors in a finite-dimensional vector space X. Show that S is a basis of X if every vector of X can be written in one and only one way as a linear combination of the vectors in S.