Read: Lax, Chapter 1, pages 1–11.

1. (a) Show that there are no proper subfields of $\mathbb{Q}$.
   (b) Show that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a field.

2. Let $X$ be a vector space over a field $K$. Let $0$ be the zero element of $K$ and $\mathbf{0}$ the zero-element of $X$. Using only the definitions of a group, a vector space, and a field, carefully prove each of the following:
   
   (a) The identity element $e$ of a group is unique.
   (b) In any group $G$, the inverse of $g \in G$ is unique.
   (c) $0x = \mathbf{0}$ for every $x \in X$;
   (d) $k\mathbf{0} = \mathbf{0}$ for every $k \in K$;
   (e) For every $k \in K$ and $x \in X$, if $kx = \mathbf{0}$, then $k = 0$ or $x = \mathbf{0}$.

3. Let $X$ denote the vector space of polynomials in $\mathbb{R}[x]$ of degree less than $n$. Are the vectors $x^3 + 2x + 5$, $3x^2 + 2$, $6x$, $6$ linearly independent in $X$? (Assume that $n \geq 4$.)

4. The following is called the Replacement Lemma: Let $X$ be a vector space over $K$, and let $S$ be a linearly independent subset of $X$. Let $x_0 \in \text{Span}(S)$ with $x_0 \neq \mathbf{0}$. Prove that there exists $x_1 \in S$ such that the set $S' = (S \setminus \{x_1\}) \cup \{x_0\}$ is a basis for $\text{Span}(S)$.
   
   (a) Prove the Replacement Lemma
   (b) Suppose that $B$ is a basis for $X$ containing $n$ elements, and let $B'$ be another basis for $X$. Show that $|B'| = n$.

5. If $Y$ is a subspace of $X$, then two vectors $x_1, x_2 \in X$ are congruent modulo $Y$, denoted $x_1 \equiv x_2 \mod Y$, if $x_1 - x_2 \in Y$. This is an equivalence relation; denote the equivalence class containing $x \in X$ by $\{x\}$, and let $X/Y$ denote the set of equivalence classes. We can make $X/Y$ into a vector space by defining addition and scalar multiplication as follows:

   \[
   \{x\} + \{z\} = \{x + z\}, \quad (ax) = a\{x\}.
   \]

   Show that these operations are well-defined, that is, they do not depend on the choice of congruence class representatives.

6. Let $S$ be a set of vectors in a finite-dimensional vector space $X$. Show that $S$ is a basis of $X$ if every vector of $X$ can be written in one and only one way as a linear combination of the vectors in $S$. 