Read: Lax, Chapter 4, pages 32–43.

1. Let $T\colon X \to U$, with $\dim X = n$ and $\dim U = m$. Show that there exist bases B for X and B' for U such that the matrix of T in block form is

$$M = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

where I_k is the $k \times k$ identity matrix, and the other blocks are either empty or contain all zeros.

2. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix representation $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ -3 & 0 & 3 \end{bmatrix}$ with respect to the standard basis. What is the matrix representation of T with respect to the basis $\{(1,-1,0),(0,1,-1),(1,0,1)\}$?

3. Let $\mathcal{P}_n$ be the vector space of all polynomials over $\mathbb{R}$ of degree less than n.

   (a) Show that the map $T\colon \mathcal{P}_3 \to \mathcal{P}_4$ given by

   $$T(p(x)) = 6\int_1^x p(t)\,dt$$

   is linear. Indicate whether it is 1–1 or onto.

   (b) Let $B_3 = \{1, x, x^2\}$ be a basis for $\mathcal{P}_3$ and let $B_4 = \{1, x, x^2, x^3\}$ be a basis for $\mathcal{P}_4$. Find the matrix representation of T with respect to these bases.

4. Find necessary and sufficient conditions on the entries u_1, u_2, u_3, u_4 under which the following system of linear equations will have at least one solution over $\mathbb{Z}_5 = \{0,1,2,3,4\}$, and give the number of solutions in case the conditions are met.

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 3 \\ 1 & -1 & -1 & 2 \\ 0 & 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}.$$

5. Consider the system of linear equations $Tx = u$, where

$$T = \begin{bmatrix} 0 & 2 & -2 \\ 3 & 5 & -2 \\ 4 & 0 & 4 \end{bmatrix}, \qquad u = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

   (a) Solve $Tx = u$ over $\mathbb{R}$.

   (b) Solve $Tx = u$ over $\mathbb{Z}_2$.