Read: Lax, Chapter 6, pages 69–76.

1. Let A be an $n \times n$ matrix over $\mathbb{C}$ with an eigenvalue λ of index $m \geq 2$ and corrresponding eigenvector v_1. Let v_2 be a generalized eigenvector satisfying $(A - \lambda I)v_2 = v_1$.

   (a) Prove that for any natural number N,
   $$A^N v_2 = \lambda^N v_2 + N\lambda^{N-1} v_1 \,.$$

   (b) Prove that for any polynomial $q(t) \in \mathbb{C}[t]$,
   $$q(A)v_2 = q(\lambda)v_2 + q'(\lambda)v_1 \,,$$
   where $q'(t)$ is the derivative of q.

   (c) Conjecture a formula for $q(A)v_m$, where $v_1, \dots, v_m$ are generalized eigenvectors of A with $(A - \lambda I)v_k = v_{k-1}$ (and say $v_0 = 0$, for convenience).

2. Consider the following matrices:
   $$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}.$$

   (a) Determine the characteristic and minimal polynomials of A, B, and C.

   (b) Determine the eigenvectors and generalized eigenvectors of A, B, and C.

3. Consider the following matrices:
   $$A = \begin{bmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

   A straightforward calculation shows that the characteristic polynomials are
   $$p_A(t) = p_B(t) = p_C(t) = (t-2)^2(t-3)\,.$$

   (a) Determine the minimal polynomials $m_A(t)$, $m_B(t)$, and $m_C(t)$.

   (b) Determine the eigenvectors and generalized eigenvectors of A, B, and C.

   (c) Determine which of these matrices are similar.

4. Let λ be an eigenvalue of A, and let N_i be the nullspace of $(A - \lambda I)^i$. Prove that $A - \lambda I$ extends to a well-defined map $N_{i+1}/N_i \longrightarrow N_i/N_{i-1}$, and that this mapping is 1–1.

5. Let A be an $n \times n$ matrix over $\mathbb{C}$.

   (a) Prove that if $A^k = A$ for some integer $k > 1$, then A is diagonalizable.

   (b) Prove that if $A^k = 0$, then $A^n = 0$.

6. Let X be an n-dimensional vector space over $\mathbb{C}$, and let $A, B\colon X \to X$ be linear maps.

   (a) Prove that if $AB = BA$, then for any eigenvector v of A with eigenvalue λ, the vector Bv is an eigenvector of A for λ.

   (b) Show that if $\{A_1, A_2, \dots \mid A_i\colon X \to X\}$ is a (possibly infinite) set of pairwise commuting maps, then there is a nonzero $x \in X$ that is an eigenvector of every A_i.

   (c) Suppose that A and B are both diagonalizable. Show that $AB = BA$ if and only if they are *simultaneously diagonalizable*, i.e., there exists an invertible $n \times n$-matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.