Read: Lax, Chapter 7, pages 77–100.

1. Prove that $\|x\| = \max\{\langle x, y \rangle : y \in K^n \text{ with } \|y\| = 1\}$.

2. Let $f$ and $g$ be continuous functions on the interval $[0, 1]$. Prove the following inequalities.

   (a) $\left( \int_0^1 f(t)g(t) \, dt \right)^2 \leq \int_0^1 f(t)^2 \, dt \int_0^1 g(t)^2 \, dt$

   (b) $\left( \int_0^1 (f(t) + g(t))^2 \, dt \right)^{1/2} \leq \left( \int_0^1 f(t)^2 \, dt \right)^{1/2} + \left( \int_0^1 g(t)^2 \, dt \right)^{1/2}$.

3. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $y_1 = (1, 2, 1, 1)$, $y_2 = (1, -1, 0, 2)$ and $y_3 = (2, 0, 1, 1)$.

4. Let $X$ be the vector space of all continuous real-valued functions on $[0, 1]$. Define an inner product on $X$ by

   $$(f, g) = \int_0^1 f(t)g(t) \, dt.$$ 

   Let $Y$ be the subspace of $X$ spanned by $f_0, f_1, f_2, f_3$, where $f_k(x) = x^k$. Find an orthonormal basis for $Y$.

5. Let $Y$ be a subspace of a Euclidean space $X$, and $P_Y : X \to X$ the orthogonal projection onto $Y$. Prove that $P_Y^* = P_Y$.

6. Show that a matrix $M$ is orthogonal iff its column vectors form an orthonormal set.

7. Let $X$ be an $n$-dimensional real Euclidean space, and $A : X \to X$ a linear map. Define the map $f : X \to X$ by $f(x, y) = x^TAy$. Give (with proof) necessary and sufficient conditions on $A$ for $f$ to be an inner product on $X$. 