Throughout, all vector spaces are finite-dimensional over a field K. You may assume that these spaces are endowed with a Euclidean structure, though this is not always needed. The set of linear maps from X to U is denoted by $\mathrm{Hom}(X, U)$.

Read: Lax, Chapter 10, pages 143–153, and Appendix 4, pages 313–316.

1. Define the *index* of a real symmetric matrix A to be the number of strictly positive eigenvalues minus the number of strictly negative eigenvalues. Suppose A and B are real symmetric matrices and $x^T Ax \leq x^T Bx$ for all $x \in X$. Prove that the index of A is at most the index of B.

2. Let A, B be self-adjoint mappings with $0 < A < B$. Here, $<$ is the partial ordering of self-adjoint mappings where $A < B$ iff $N - A$ is positive-definite.
   (a) Show by example that the *symmetrized product* $S := AB + BA$ need not be positive-definite.
   (b) Show that $A^{1/k} < B^{1/k}$ if k is a power of 2.
   (c) Show that $\log A \leq \log B$.
   (d) Show by example that $A^2 < B^2$ need not hold.

3. Let U and V be vector spaces over a field K. Define the map
$$\varphi\colon U \otimes V \longrightarrow \mathrm{Hom}(U', V)\,, \qquad \varphi\colon u \otimes v \longmapsto \{\ell \mapsto (\ell, u)v\}\,,$$
and extend linearly to arbitrary elements $\sum(u_i \otimes v_i)$ in $U \otimes V$. Prove that φ is an isomorphism.

4. Let U, V, and X by vector spaces over a field K. Define a map
$$\iota\colon U \times V \longrightarrow U \otimes V\,, \qquad \iota(u, v) = u \otimes v\,.$$
   (a) Prove that ι is bilinear.
   (b) Prove that if $A \in \mathrm{Hom}(U \otimes V, X)$, then $\alpha := A \circ \iota$ is a bilinear map from $U \times V$ to X.
   (c) Prove that for any bilinear map $\tau\colon U \times V \to X$, there is a unique $A \in \mathrm{Hom}(U \otimes V, X)$ such that $\alpha = A \circ \iota$. This is the *universal property* of the tensor product, and is summarized by the following commutative diagram:
$$\begin{tikzcd} U \times V \arrow[rr, "\tau"] \arrow[dr, "\iota"'] & & X \\ & U \otimes V \arrow[ur, dashed, "A"'] & \end{tikzcd}$$

5. Use the universal property of the tensor product to prove the following results:
   (a) $U \otimes V \cong V \otimes U$ (hint: let $X = V \otimes U$);
   (b) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$;
   (c) $(U \times V) \otimes W \cong (U \otimes W) \times (V \otimes W)$.