Math 8530: Linear Algebra

4. Matrices

Let $T : X \to U$ be a linear map.

Goal: Encode T as a matrix.

We need to pick:
- a basis $\mathcal{B}_{in} = \{x_1, \ldots, x_n\}$ for X ("input basis")
- a basis $\mathcal{B}_{out} = \{u_1, \ldots, u_m\}$ for U ("output basis")

Let $\{l_1, \ldots, l_m\}$ be the dual basis in U'.

Next, write
$$\begin{aligned}
Tx_1 &= a_{11}u_1 + a_{21}u_1 + \cdots + a_{m1}u_m \\
Tx_2 &= a_{12}u_1 + a_{22}u_2 + \cdots + a_{m2}u_m \\
&\vdots \\
Tx_j &= a_{ij}u_1 + \cdots + a_{ij}u_i + \cdots + a_{mj}u_m \\
&\vdots \\
Tx_n &= a_{1n}u_1 + a_{2n}u_n + \cdots + a_{mn}u_m
\end{aligned}$$

$$\boxed{a_{ij} = l_i(Tx_j) = (l_1, Tx_j)}$$

The matrix of T wrt $\mathcal{B}_{in}$ and $\mathcal{B}_{out}$ is

$$A = {}_{\mathcal{B}_{in}}[T]_{\mathcal{B}_{out}} := \begin{array}{c}\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{21} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ \begin{array}{cccc} Ax_1 & Ax_2 & & A_n \end{array}\end{array}$$

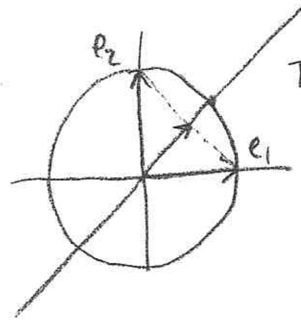
Remarks
- $R_T = \text{Span(col. vectors)}$, also called the "column space."
- $a_{ij} = (l_i, Tx_j)$

Ex: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the line $y = x$.

First, consider $\mathcal{B}_{in} = \mathcal{B}_{out} = \{e_1, e_2\}$.

$\boxed{2}$

$$Te_1 = \tfrac{1}{2} e_1 + \tfrac{1}{2} e_2$$
$$Te_2 = \tfrac{1}{2} e_1 + \tfrac{1}{2} e_2$$

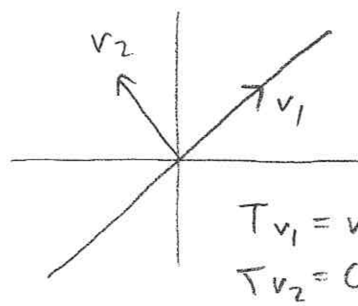


$Te_1 = Te_2 = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$

So $$ {}_{\mathcal{B}_{in}}[T]_{\mathcal{B}_{out}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

Let's pick a different basis: $\mathcal{B}'_{in} = \mathcal{B}'_{out} = \left\{ v_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, v_2 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \right\}$.

$$Tv_1 = 1\,v_1 + 0\,v_2$$
$$Tv_2 = 0\,v_1 + 0\,v_2 \quad \Rightarrow \quad {}_{\mathcal{B}'_{in}}[T]_{\mathcal{B}'_{out}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



<u>Remark</u>: If $T : X \to X$ is invertible, then we can <u>always</u> choose $\mathcal{B}_{in}$ and $\mathcal{B}_{out}$ so ${}_{\mathcal{B}_{in}}[T]_{\mathcal{B}_{out}} = I$,

If $\mathcal{B}_{in} = \{v_1, \dots, v_n\}$,

then $\mathcal{B}_{out} := \{Tv_1, \dots, Tv_n\}$.

<u>Ex</u>: Let $X = \mathcal{P}_2 = \{ c_0 + c_1 x + c_2 x^2 : c_i \in \mathbb{R} \}$ $\quad \mathcal{B}_{in} = \{1, x, x^2\}$

$U = \mathcal{P}_1 = \{ c_0 + c_1 x : c_i \in \mathbb{R} \}$ $\quad \mathcal{B}_{out} = \{1, x\}$

Let $T = \frac{d}{dx}$, so $T(c_0 + c_1 x + c_2 x^2) = c_1 + 2c_2 x$.

$$T(1) = 0 \cdot 1 + 0\,x$$
$$T(x) = 1 \cdot 1 + 0\,x$$
$$T(x^2) = 0 \cdot 1 + 2\,x$$

$$A = {}_{\mathcal{B}_{in}}[T]_{\mathcal{B}_{out}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Multiplying matrices (4 ways)

(1) Rows times columns

$$\text{row } i \begin{bmatrix} & & \\ \text{---} & l_i & \text{---} \\ & & \end{bmatrix}_{m\times n} \begin{array}{c} x_j \\ \begin{bmatrix} & | & \\ & | & \\ & | & \end{bmatrix}_{n\times p} \\ \text{col } j \end{array} = \begin{bmatrix} & & \\ & \overset{a_{ij}}{\bullet} & \\ & & \end{bmatrix}_{m\times p}$$

$$\boxed{a_{ij} = (l_i, x_j)}$$

(2) By columns:

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \overset{B}{\begin{bmatrix} | & & | \\ | & \cdots & | \\ \vec{x}_1 & \cdots & \vec{x}_p \end{bmatrix}} = \underbrace{\begin{bmatrix} | & & | \\ | & \cdots & | \\ A\vec{x}_1 & & A\vec{x}_p \end{bmatrix}}_{\text{these columns are linear combinations of cols. of } A.}$$

(3) By rows

$$\begin{bmatrix} \text{---}\ l_1\ \text{---} \\ \vdots \\ \text{---}\ l_m\ \text{---} \end{bmatrix} \begin{bmatrix} \\ B \\ \\ \end{bmatrix} = \begin{bmatrix} \text{---}\ l_1 B\ \text{---} \\ \vdots \\ \text{---}\ l_m B\ \text{---} \end{bmatrix}$$

Each row is a linear combin. of rows of B.

(4) Sum of rank 1 matrices: (cols of A) × (rows of B)

$$\underset{n \text{ columns}}{\overset{A}{\begin{bmatrix} | & & | \\ | & \cdots & | \end{bmatrix}}} \underset{n \text{ rows}}{\overset{B}{\begin{bmatrix} \text{---} \\ \vdots \\ \text{---} \end{bmatrix}}} = \sum_{j,k=1}^{n} \underbrace{\vec{a}_j \vec{b}_k^T}_{m\times p \text{ matrix}}$$

Recall: If $T: X \to U$ is linear, and A is the matrix wrt bases $x_1, \dots, x_n$ & $\underbrace{u_1, \dots, u_m}$

with dual basis $l_1, \dots, l_m \in U'$.

[that is, $l_i(u_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$]

then $\boxed{a_{ij} = (l_i, Tx_j)}$

Thus, to get a_{ij}, where $A = (a_{ij})$:

- Apply T to the j^{th} basis vector of X
- Then apply the i^{th} co-vector in U' to this

Now, consider $T': U' \to X'$. Find matrix (a_{ij}')

- Apply T' to the j^{th} basis vector of U'
- Then apply the i^{th} co-vector in $X'' = X$ to this:

$$a_{ij}' = (\hat{x}_i, T'l_j) \Rightarrow (T'l_j, x_i) = (l_j, Tx_i) = a_{ji}$$

in X'' ↑ ↑ identify $X'' = X$

* Thus, the matrix (a_{ij}') for T' satisfies $a_{ij}' = a_{ji}$.

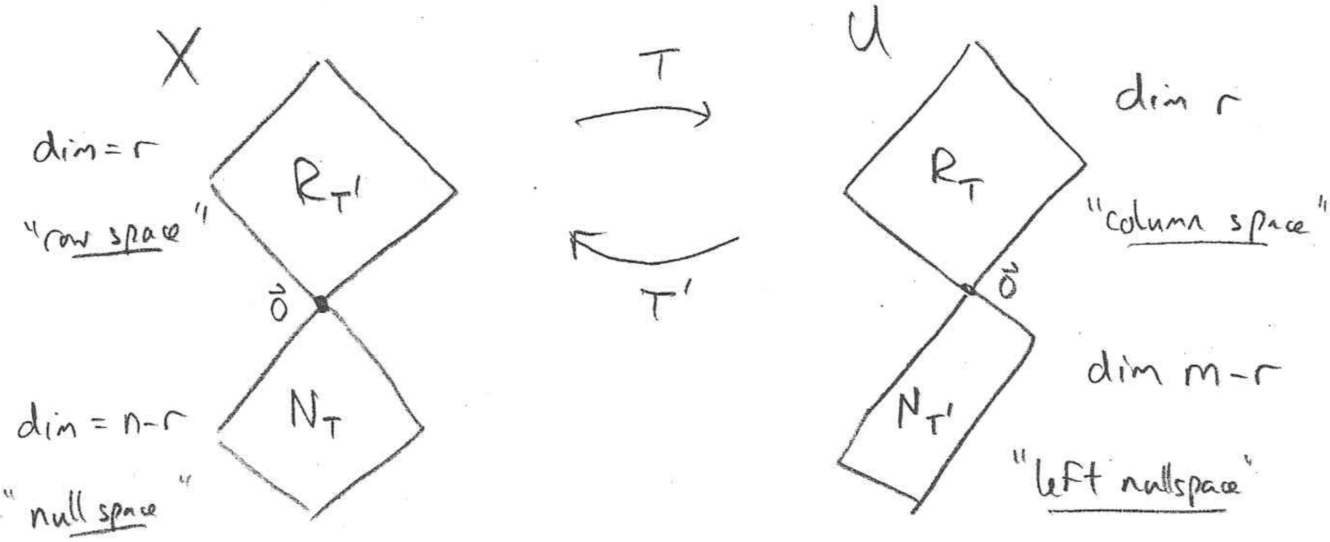
Denote this matrix by A^T. Obtained by swapping rows with columns.

<u>Recall</u>: The range R_T consists of all linear combinations of the column vectors. The dimension of this space is called the <u>column rank</u> of T. The <u>row rank</u> is defined similarly.

<u>Remarks</u>: (i) row rank of T is $\dim R_{T'}$

(ii) By thm 3.6 ($\dim R_T = \dim R_{T'}$), row rank and column rank are equal.

"<u>Cartoon</u>" of this: $T : X \to U$



<u>Prop</u>: T is a bijection when restricted $R_{T'} \to R_T$.

<u>Proof</u>: Only need to show it's 1-1.

Suppose $T\ell_1 = T\ell_2$. $\Rightarrow T(\ell_1 - \ell_2) = 0 \Rightarrow \ell_1 - \ell_2 \in N_T$

But $\ell_1 - \ell_2$ also in $R_{T'}$ (since its a subspace)

$\Rightarrow \ell_1 - \ell_2 = 0$, since $X = R_{T'} \oplus N_T$. $\square$

Change of basis

Let $T : X \to U$ be linear, $x_1, \dots, x_n$ & $u_1, \dots, u_m$ be bases.

Since $\dim X = n$, $\dim U = m$, we have $X \cong \mathbb{R}^n$, $U \cong \mathbb{R}^m$.

That is, we have isomorphisms:

$$B : X \longrightarrow \mathbb{R}^n \qquad C : U \longrightarrow \mathbb{R}^m$$

$$x_i \longmapsto e_i \qquad u_i \longmapsto e_i$$

Putting this together, we can choose isomorphisms such as these to get a linear map $CTB^{-1} : \mathbb{R}^n \to \mathbb{R}^m$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "B"'] & U \arrow[d, "C"] \\ \mathbb{R}^n \arrow[r, "CTB^{-1}"] & \mathbb{R}^m \end{tikzcd}$$

If $T : X \to X$ then we can take $x_i = u_i$ & $B = C$ and get a matrix $M = BTB^{-1}$.

Suppose we change the isomorphism B. In particular, let $A : X \to \mathbb{R}^n$ be another isomorphism, and let N be the matrix wrt this basis, i.e., $N = ATA^{-1}$

We have $N = ATA^{-1} = AB^{-1}BTB^{-1}BA^{-1} = SMS^{-1}$,

where $S = AB^{-1}$, which is invertible:

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "B"'] & X \arrow[d, "B"] \\
\mathbb{R}^n \arrow[r, "M = BTB^{-1}"] \arrow[d, "B^{-1}"'] \arrow[dd, bend right, "S"'] & \mathbb{R}^n \arrow[d, "B^{-1}"] \arrow[dd, bend left, "S"] \\
X \arrow[r] \arrow[d, "A"'] & X \arrow[d, "A"] \\
\mathbb{R}^n \arrow[r, "N = SMS^{-1}"] & \mathbb{R}^n
\end{tikzcd}$$

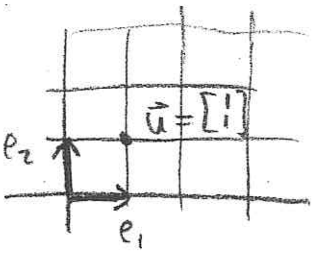
Two square matrices M, N related by conjugation (e.g., $N = SMS^{-1}$) are said to be <u>conjugate</u>.

* <u>Similar matrices</u> describe the <u>same</u> mapping of a space into itself, but using <u>different bases</u>. Thus (as we'll see later) similar matrices share the same intrinsic properties.

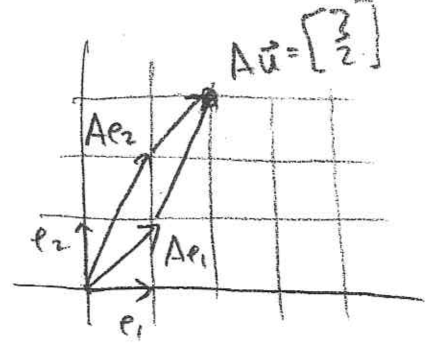
<u>Example:</u> Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

<u>Input basis:</u>

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



Say $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$\downarrow M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$Mu = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

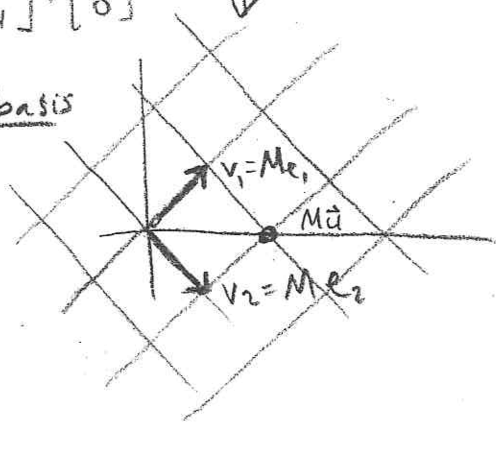
<u>Output basis:</u>

e_1, e_2

AM^{-1}

<u>New input basis</u>

$v_1 = Me_1$
$v_2 = Me_2$



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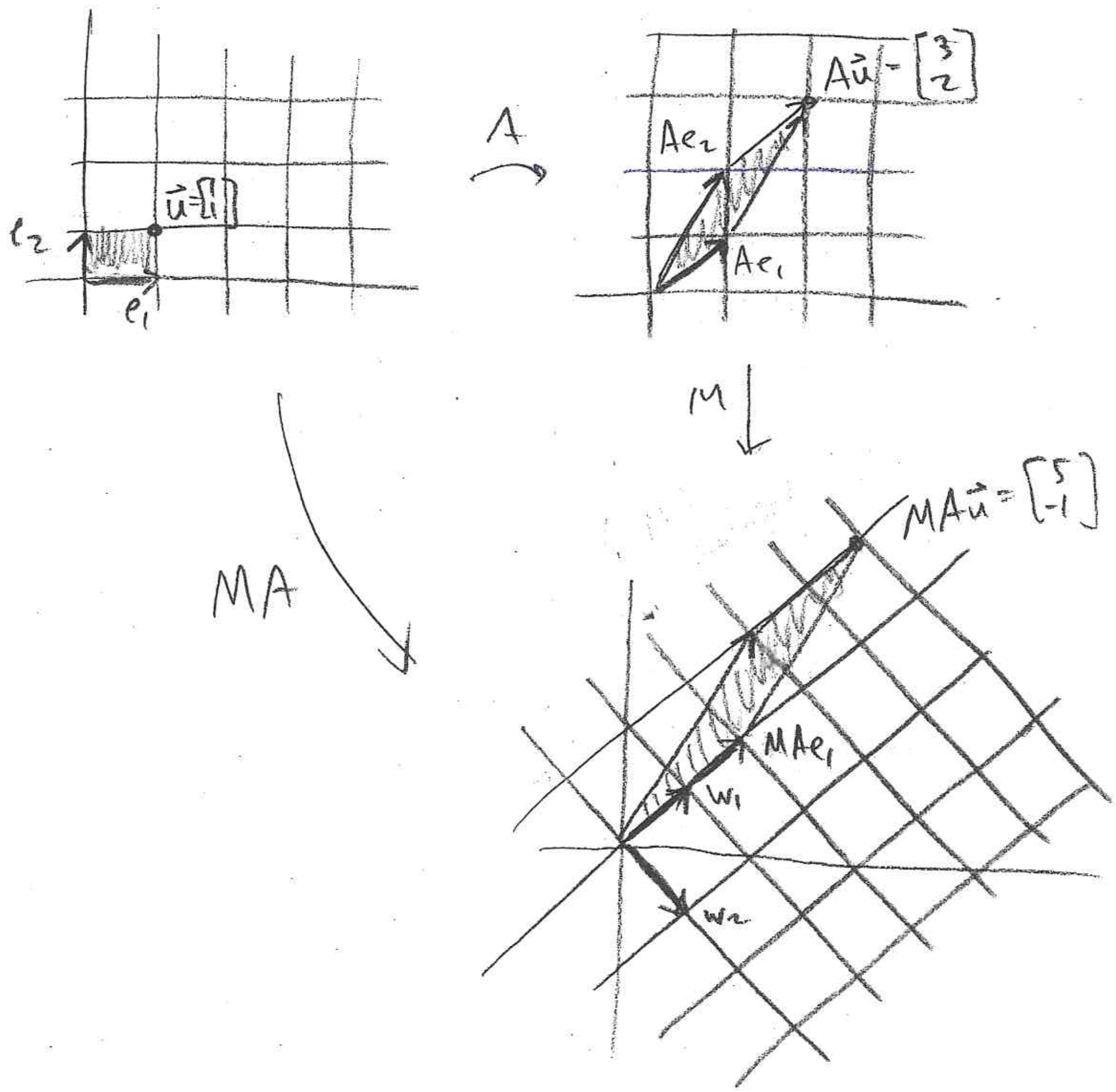
Now, let's change the output basis instead



Output basis: $w_1 = Me_1$
$w_2 = Me_2$

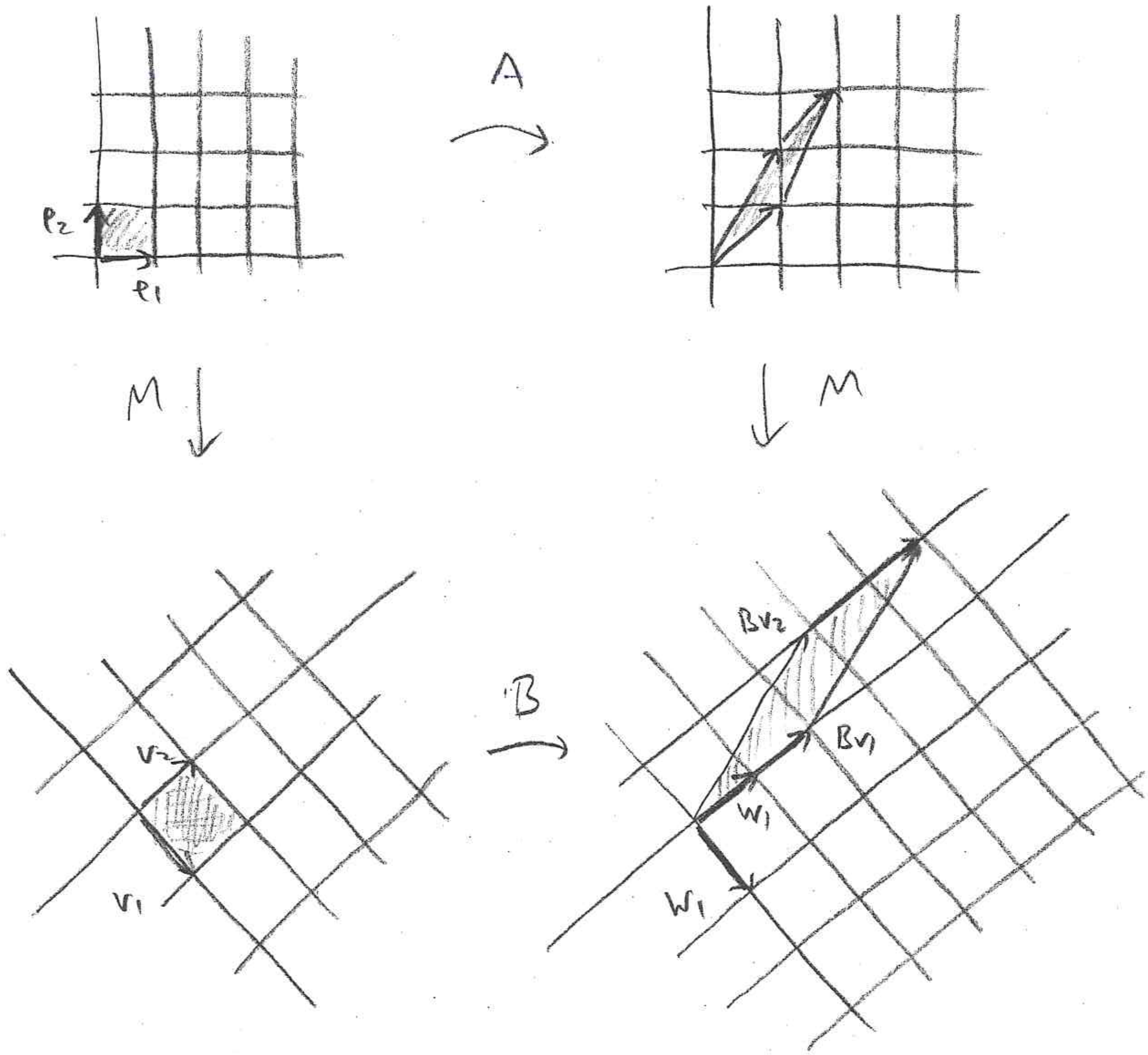
$$MA\vec{u} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2 & 3\\0 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5\\-1\end{bmatrix}$$

In our new coordinates: $MA\vec{e}_1 = \begin{bmatrix}2 & 3\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$

$MA\vec{e}_2 = \begin{bmatrix}2 & 3\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix}$.

Now, let's change <u>both</u> input & output bases:

$$v_1 = w_1 = Me_1 \quad , \quad v_2 = w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



Note that $A = M^{-1} B M$

$$B = M A M^{-1}$$

<u>Remark</u>: We can write any $n\times n$ matrix A in <u>block form</u>:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{where} \quad A = \underbrace{\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right)}_{k \text{ columns} \quad n-k \text{ columns}} \begin{array}{l} \} \, k \text{ rows} \\ \} \, n-k \text{ rows} \end{array}$$

Addition and multiplication of block matrices "works out" just as if the blocks were entries.

<u>Def</u>: The square matrix $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is called the <u>identity</u> or <u>unit</u> matrix.

<u>Def</u>: A square matrix $T = (t_{ij})$ for which $t_{ij} = 0$ for $i > j$ (resp. $i < j$) is called <u>upper triangular</u> (resp. lower triangular).

<u>Systems of equations</u>

Matrices can be used to effectively express and solve systems of equations.

A system $\sum_{i=1}^{n} t_{ij} x_i = u$ for $j = 1, \dots, n$ may have a unique sol'n, many sol'ns, or no solutions.

Example: $x_1, \ldots, x_4$ are unknowns:

$$
\begin{aligned}
x_1 + x_2 + 2x_3 + 3x_4 &= u_1 \\
x_1 + 2x_2 + 3x_3 + x_4 &= u_2 \\
2x_1 + x_2 + 2x_3 + 3x_4 &= u_3 \\
3x_1 + 4x_2 + 6x_3 + 2x_4 &= u_4
\end{aligned}
\quad \rightsquigarrow \quad
\underbrace{\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 3 \\ 3 & 4 & 6 & 2 \end{pmatrix}}_{M}
\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{\vec{x}}
=
\underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}}_{\vec{u}}
$$

System can be solved by Gaussian elimination:

Subtract multiples of Eq 1 from last 3 eq'ns to eliminate x_1.

$$
\begin{aligned}
x_2 + x_3 - 2x_4 &= u_2 - u_1 \\
-x_2 - 2x_3 - 3x_4 &= u_3 - 2u_1 \\
x_2 \quad\quad - 7x_4 &= u_4 - 3u_1
\end{aligned}
$$

Use 1st eqn to eliminate x_2 from last 2:

$$
\begin{aligned}
-x_3 - 5x_4 &= u_3 + u_2 - 3u_1 \\
-x_3 - 5x_4 &= u_4 - 2u_2 - 2u_1
\end{aligned}
$$

Subtract these equations to eliminate x_3 (& by chance, x_4 too).

$$\boxed{0 = u_4 - u_3 - 2u_2 + u_1} :$$

This is a necessary & sufficient condition for our original system to have a sol'n. It can be written as

$(l, u) = 0$, where $l = (1, -2, -1, 1)$

That is, $(1, -2, -1, 1)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = u_1 - 2u_2 - u_3 + u_4 = 0$

Since $Mx = u$ and $lu = 0$, we must have

$$lMx = lu = 0 \text{ for all } x \in \mathbb{R}^4, \text{ thus } lM = 0.$$

* In general, if $Mx = u$ is a system of equations, then a sol'n is described by a linear function $l \in X'$, or equivalently, a column vector for which $lM = 0$.