Read: Strang, Section 1.1, 1.2, and 2.1.

1. For each vector $\boldsymbol{v}$, compute its *norm*, $||\boldsymbol{v}|| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}$, and then *normalize* it, by computing $\boldsymbol{v}/||\boldsymbol{v}||$.

$$\boldsymbol{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

2. For a unit vector $\boldsymbol{n}$, the *projection* of $\boldsymbol{v}$ onto $\boldsymbol{n}$ is the quantity $\boldsymbol{v}\cdot\boldsymbol{n}$. This measures the magnitude of $\boldsymbol{v}$ in the $\boldsymbol{n}$-direction. Consider the following four unit vectors:

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{n}_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, \qquad \boldsymbol{n}_2 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}.$$

   (a) Draw the vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ in $\mathbb{R}^2$, and sketch the square "grid" that they determine. Do the same thing for $\{\boldsymbol{n}_1, \boldsymbol{n}_2\}$ but on a new set of axes.

   (b) Write the vector $\boldsymbol{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ as a *linear combination* of $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. That is, write $\boldsymbol{w} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2$ and determine $a_1, a_2 \in \mathbb{R}$. Then, write $\boldsymbol{w}$ as a linear combination of $\{\boldsymbol{n}_1, \boldsymbol{n}_2\}$.

   (c) Sketch $\boldsymbol{w}$ on both sets of axes, and show how these sketches match your answers to Part (b).

   (d) The 2×2 matrix $\boldsymbol{A} = [\boldsymbol{n}_1 \ \ \boldsymbol{n}_2]$ can be thought of as a *linear map*, $\boldsymbol{A}\colon \mathbb{R}^2 \to \mathbb{R}^2$. Describe this linear map (geometrically) in a sentence. [*Hint*: You can think of the "input" as one of your grids, and the "output" as the other grid.]

3. A set $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ of vectors is *orthogonal* (or *perpendicular*) if $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$ for all $i \neq j$. The set is furthermore *orthonormal* if each $\boldsymbol{v}_i$ is a unit vector. That is, if

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

   (a) Show that the set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}.$$

   is an orthogonal set, but not orthonormal.

   (b) Normalize $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ to get an orthonormal *basis* of $\mathbb{R}^3$, $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\}$.

   (c) Express the vector $\boldsymbol{w} = (1, 2, 3)$ in terms of $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$. That is, find a_1, a_2, and a_3 such that

$$\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a_1\boldsymbol{n}_1 + a_2\boldsymbol{n}_2 + a_3\boldsymbol{n}_3\,.$$

(d) Express the vector $\boldsymbol{w}$ as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$. That is, write $\boldsymbol{w} = b_1\boldsymbol{v}_1 + b_2\boldsymbol{v}_2 + b_3\boldsymbol{v}_3$ and find a formula for each b_i. [*Hint*: It should be in entirely in terms of dot products of $\boldsymbol{v}_i$ and $\boldsymbol{w}$. Start by substituting $\boldsymbol{v}_i/||\boldsymbol{v}_i||$ in for $\boldsymbol{n}_i$ in your answer to Part (c).]

4. Consider the matrix $\boldsymbol{A} = [\boldsymbol{u} \;\; \boldsymbol{v} \;\; \boldsymbol{w}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

(a) Find a non-zero linear combination $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}$ of the column vectors of $\boldsymbol{A}$ that gives the zero vector.

(b) Describe the set of all *solutions* to $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w} = \boldsymbol{0}$. [*Hint*: It is a line. Which line is it?]

(c) Describe the set of all *linear combinations* $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}$. We say that this is the *subspace* of $\mathbb{R}^3$ that is *spanned* by the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$.

5. Given an $n \times m$ matrix $\boldsymbol{A} = [a_{ij}]$, the *transpose* of $\boldsymbol{A}$ is an $m \times n$ matrix defined as $\boldsymbol{A}^T = [a_{ji}]$. For each of the following three matrices $\boldsymbol{M}$, compute its transpose $\boldsymbol{M}^T$, as well as the products $\boldsymbol{M}^T\boldsymbol{M}$ and $\boldsymbol{M}\boldsymbol{M}^T$.

$$\boldsymbol{A} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 0 & 4 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. Consider the following vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, and the matrix $\boldsymbol{A} = [\boldsymbol{u} \;\; \boldsymbol{v} \;\; \boldsymbol{w}]$:

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}.$$

(a) Write out $\boldsymbol{A}^T\boldsymbol{A}$ in terms of the *dot products* of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

(b) Write out $\boldsymbol{A}^T\boldsymbol{A}$ in terms of the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ and their transposes, but *not* their individual entries. [Recall that $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T\boldsymbol{v}$.]

(c) Give a complete characterization of which matrices $\boldsymbol{A}$ have the property that $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the *identity matrix*. Give a geometric description of your answer in terms of the column vectors.