*Read*: Strang, Section 1.3, 2.1, 2.2, 2.3.

1. In this problem, we will solve a system $\boldsymbol{Ax} = \boldsymbol{b}$ using the method of *elimination*, also called *row-reduction*. The system is shown in matrix form on the left, and the corresponding *augmented matrix* is shown at right, which is a useful notation.

$$\begin{bmatrix} 1 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad\qquad [\boldsymbol{A} \mid \boldsymbol{b}] = \left[\begin{array}{cc|c} 1 & -2 & 3 \\ -4 & 2 & 0 \end{array}\right].$$

Carry out the following steps for this system.

   (a) Perform a single elimination step to convert the equation $\boldsymbol{Ax} = \boldsymbol{b}$ into an equation $\boldsymbol{Ux} = \boldsymbol{c}$, where $\boldsymbol{U}$ is an *upper triangular matrix*. What is the corresponding elementary matrix, $\boldsymbol{E}_{21}$? Notice that $\boldsymbol{U} = \boldsymbol{E}_{21}\boldsymbol{A}$.

   (b) Now, perform a single elimination step to remove the entry of $\boldsymbol{U}$ *above* the main diagonal. You will end up with a equation of the form $\boldsymbol{Dx} = \boldsymbol{d}$, where $\boldsymbol{D}$ is a *diagonal matrix*.

   (c) What is $\boldsymbol{D}^{-1}$, the *inverse matrix* of $\boldsymbol{D}$? (You should be able to do this by inspection.) Multiply both sides of the equation $\boldsymbol{Dx} = \boldsymbol{d}$ by $\boldsymbol{D}^{-1}$ to solve for $\boldsymbol{x}$.

   (d) The solution you obtained in Part (c) is actually the vector $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$. Express $\boldsymbol{A}^{-1}$ as a product of matrices that you found above ($\boldsymbol{E}_{12}$, $\boldsymbol{E}_{21}$, and $\boldsymbol{D}^{-1}$).

2. Consider the following $3 \times 3$ system, $\boldsymbol{Ax} = \boldsymbol{b}$. The augmented matrix $[\boldsymbol{A} \mid \boldsymbol{b}]$ is not shown.

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \\ -2 & -7 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

   (a) Perform a sequence of elimination steps to convert the equation $\boldsymbol{Ax} = \boldsymbol{b}$ into an equation $\boldsymbol{Ux} = \boldsymbol{c}$, where $\boldsymbol{U}$ is an *upper triangular matrix*. For each step, write down the corresponding elementary matrix, $\boldsymbol{E}_{ij}$.

   (b) Express $\boldsymbol{U}$ as a product of elementary matrices and $\boldsymbol{A}$. (Note that $\boldsymbol{A}$ should be the rightmost matrix in the expression.)

   (c) Continue the elimination process to eliminate the entries of $\boldsymbol{U}$ above the main diagonal. You will end up with a equation of the form $\boldsymbol{Dx} = \boldsymbol{d}$, where $\boldsymbol{D}$ is a diagonal matrix.

   (d) Find $\boldsymbol{D}^{-1}$ and multiply it to both sides of the equation $\boldsymbol{Dx} = \boldsymbol{d}$ to solve for $\boldsymbol{x}$.

   (e) Note that the solution you obtained in Part (d) is $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$. Express $\boldsymbol{A}^{-1}$ as a product of elementary matrices and $\boldsymbol{D}^{-1}$.

3. In both of the following equations, the matrix $\boldsymbol{A}$ is not invertible, so there will either be no solution, or infinitely many solutions to the equation $\boldsymbol{Ax} = \boldsymbol{b}$.

$$\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \qquad\qquad \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

Carry out the following steps for each equation above.

(a) Graph the two linear equations of the system (this is the "row picture").

(b) Use the eliminaton method until you get an unavoidable zero in a pivot location, in which case you have to stop. At this point you will have an equation $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{c}$, where $\boldsymbol{U}$ is upper-triangular.

(c) Write out the two equations that the system $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{c}$ represents. Are there no solutions, or infinitely many solutions? Why?

4. We will solve the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ for each of the three matrices $\boldsymbol{A}$ shown below.

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 5 & 6 \\ 5 & -4 & 10 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 3 & -6 & 3 \end{bmatrix}$$

Recall that if $\boldsymbol{A}$ is not invertible, then there will be a non-zero solution to the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. Carry out the following steps for each equation above.

(a) Use the elimination method to row-reduce $\boldsymbol{A}$ to an upper-triangular matrix $\boldsymbol{U}$, and if possible, to a diagonal matrix $\boldsymbol{D}$. (Show your steps, but you do not need to write out each individual elementary matrix like you did in Problems 1 and 2.)

(b) Describe the set of all solutions to $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{0}$ (and hence to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$) as best you can. Give a geometric interpretation (e.g., a point, line, plane, etc.)

5. Find the inverse of the following matrices by row-reducing the corresponding *augmented matrix* until the left half is the identity matrix.

$$[\boldsymbol{A} \mid \boldsymbol{I}] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 3 & -7 & 0 & 1 \end{array}\right] \qquad [\boldsymbol{B} \mid \boldsymbol{I}] = \left[\begin{array}{ccc|ccc} 3 & 1 & 4 & 1 & 0 & 0 \\ 5 & 2 & 6 & 0 & 1 & 0 \\ -7 & -2 & -9 & 0 & 0 & 1 \end{array}\right]$$

When you do this, the augmented matrices that you end up with will be $[\boldsymbol{I} \mid \boldsymbol{A}^{-1}]$ and $[\boldsymbol{I} \mid \boldsymbol{B}^{-1}]$, resepectively. [*Hint*: The matrix $\boldsymbol{B}^{-1}$ has only integer entries – by scaling appropriately, you can avoid fractions altogether in the elimination process.]