Read: Strang, Section 4.1, 4.2.

Suggested short conceptual exercises: Strang, Section 4.1, #1, 2, 4, 5, 8–10, 13, 15, 18,–21, 24–29. Section 4.2, #13, 18, 21–28.

1. Construct a nonzero matrix $\boldsymbol{A}$ with the required property or say why it is impossible:
   (a) The column space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, and the nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
   (b) The row space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, and the nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
   (c) $\boldsymbol{Ax} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution and $\boldsymbol{A}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
   (d) Every row is orthogonal to every column.
   (e) The sum of the columns is the zero vector, and the sum of the rows is a vector with all 1's.

2. Consider the following system of equations $\boldsymbol{Ax} = \boldsymbol{b}$:

$$\begin{aligned} x + 2y + 2z &= b_1 \\ 2x + 2y + 3z &= b_2 \\ 3x + 4y + 5z &= b_3. \end{aligned}$$

   (a) Find numbers y_1, y_2, y_3 to multiply the left-hand sides of the equations so they add to 0. You have found a vector $\boldsymbol{y}$ in which subspace? Write $\boldsymbol{y}^T\boldsymbol{b} = 0$ in terms of b_1, b_2, and b_3?
   (b) Using orthogonality of subspaces, what must be the case about $\boldsymbol{y}$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ for there to be a solution to $\boldsymbol{Ax} = \boldsymbol{b}$? Does this condition hold for $\boldsymbol{b} = (5, 5, 9)$?
   (c) What happens when we left-multiply both sides of the equation $\boldsymbol{Ax} = (5, 5, 9)$ by $\boldsymbol{y}^T$, where $\boldsymbol{y}$ is from Part (a)?

3. For each matrix, accurately sketch the four fundamental subspaces on two $\mathbb{R}^2$ plots so that orthogonal pairs of subspaces are plotted together. This is the "grid picture."

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

4. For a set S, let $S^\perp$ denote the *orthogonal complement* of S, i.e., the set of vectors orthogonal to all vectors in S. Note that even if S is not a subspace, $S^\perp$ is.
   (a) If S is the subspace of $\mathbb{R}^3$ containing only the zero vector, what is $S^\perp$? (Find a basis.)

(b) If S is spanned by $(1,1,1)$, what is $S^\perp$? (Find a basis.)

(c) If S is spanned by $(1,1,1)$ and $(1,1,-1)$, what is a basis for $S^\perp$? (Find a basis.)

(d) Now, suppose S is not a subspace, but rather just the set containing the two vectors $(1,1,1)$ and $(1,1,-1)$. What is $S^\perp$? It is the nullspace of what matrix?

(e) Suppose S is a set of vectors (not necessarily a subspace). Describe as concisely as possible what subspace $(S^\perp)^\perp$ is. What is the relation between S and $(S^\perp)^\perp$ when S actually is a subspace.

5. Let $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   (a) Project $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$. Check that $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is orthogonal to $\boldsymbol{a}$.

   (b) Find the projection matrix $\boldsymbol{P} = \dfrac{\boldsymbol{a}\boldsymbol{a}^T}{\boldsymbol{a}^T\boldsymbol{a}}$ onto the line through $\boldsymbol{a}$. Verify that $\boldsymbol{P}^2 = \boldsymbol{P}$. Multiply $\boldsymbol{P}\boldsymbol{b}$ to compute the projection $\boldsymbol{p}$.

6. Let $\boldsymbol{a}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

   (a) Compute the projection matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ onto the lines through $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Multiply those matrices and explain geometrically why $\boldsymbol{P}_1\boldsymbol{P}_2$ is what it is.

   (b) Project $\boldsymbol{b} = (1,0,0)$ onto the lines through $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ and also onto $\boldsymbol{a}_3$. Add up the three projections $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3$.

   (c) Find the projection matrix $\boldsymbol{P}_3$ onto $\boldsymbol{a}_3$. Verify that $\boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3 = \boldsymbol{I}$. This means that the basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ is orthogonal! (Think about why.)

7. Suppose $\boldsymbol{P}$ is a projection matrix onto the column space of $\boldsymbol{A}$.

   (a) Show that the matrix $\boldsymbol{I} - \boldsymbol{P}$ is also a projection matrix by verifying that $(\boldsymbol{I} - \boldsymbol{P})^T = \boldsymbol{I} - \boldsymbol{P}$ and $(\boldsymbol{I} - \boldsymbol{P})^2 = \boldsymbol{I} - \boldsymbol{P}$ both hold.

   (b) What subspace does the matrix $\boldsymbol{I} - \boldsymbol{P}$ project onto? [*Hint*: Note that $\boldsymbol{b} = \boldsymbol{P}\boldsymbol{b} + (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{b}$ holds for any vector $\boldsymbol{b}$!]

8. Consider the plane $\mathcal{P}$ in $\mathbb{R}^3$ given by $x - y - 2z = 0$.

   (a) Find a matrix whose columns are a basis for $\mathcal{P}$.

   (b) Compute $\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$, which is the projection matrix onto $\mathcal{P}$.

   (c) Find a vector $\boldsymbol{e}$ that is orthogonal to $\mathcal{P}$. Compute the projection matrix $\boldsymbol{Q} = \boldsymbol{e}\boldsymbol{e}^T / \boldsymbol{e}^T\boldsymbol{e}$ and $\boldsymbol{I} - \boldsymbol{Q}$. How are $\boldsymbol{P}$ and $\boldsymbol{Q}$ related?