Read: Strang, Section 4.3, 4.4.

Suggested short conceptual exercises: Strang, Section 4.3, #12–16, 25, 26, 29. Section 4.4, #3, 4, 8, 9, 19.

1. For this problem, consider the four data points $(t_i, b_i) = (0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$. Let $\boldsymbol{t} = (0,1,3,4)$ be the vector of inputs and $\boldsymbol{b} = (0,8,8,20)$ the vector of outputs. Feel free to use a computer to solve *any* systems of equations you encounter throughout this problem.

   (a) If there were a straight line $b = C + Dt$ through these four points, then a certain equation $\boldsymbol{Ax} = \boldsymbol{b}$ would have a solution, where $\boldsymbol{x} = (C, D)$. Write this equation in matrix form (that is, find $\boldsymbol{A}$).

   (b) Instead, we wish to find the "best fit" line, which means we need to solve $\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{p}$, where $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$. Write down the *normal equations* $\boldsymbol{A}^T\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b}$, where $\widehat{\boldsymbol{x}} = (\widehat{C}, \widehat{D})$, and solve for $\widehat{\boldsymbol{x}}$.

   (c) Check that $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is orthogonal to both columns of $\boldsymbol{A}$. Compute $||\boldsymbol{e}||$, which is the shortest distance from $\boldsymbol{b}$ to the column space of $\boldsymbol{A}$. Sketch a diagram of $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{p}$, and the orthogonal subspaces $C(\boldsymbol{A})$ and $N(\boldsymbol{A}^T)$ to illustrate this.

   (d) Plot the four data points in $\mathbb{R}^2$ (on the tb-plane) and sketch the best fit line through them that you just found. Clearly mark what the vectors $\boldsymbol{b} = (b_1, b_2, b_3, b_4)$, $\boldsymbol{e} = (e_1, e_2, e_3, e_4)$, and $\boldsymbol{p} = (p_1, p_2, p_3, p_4)$ represent.

   (e) Write down $E := ||\boldsymbol{Ax} - \boldsymbol{b}||^2$ as a sum of four squares–the last one is $(C + 4D - 20)^2$, and compute $\partial E/\partial C$ and $\partial E/\partial D$. Set these derivatives equal to zero, and obtain scalars of the normal equations $\boldsymbol{A}^T\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b}$.

   (f) The method above found the best fit degree-1 polynomial (line). Now, find the best fit degree-0 polynomial (horizontal line) $b = C$. Note that this will be a 4×1 system instead of a 4×2 system. Compute the vectors $\boldsymbol{p}$ and $\boldsymbol{e}$, and the (squared) error $||\boldsymbol{e}||^2$.

   (g) Find the best fit parabola (degree-2 polynomial) $b = C + Dt + Et^2$. On a new set of axes, plot the four data points and this parabola. Compute the vectors $\boldsymbol{p}$ and $\boldsymbol{e}$, and the (squared) error $||\boldsymbol{e}||^2$.

   (h) Find the best fit cubic (degree-3 polynomial) $b = C + Dt + Et^2 + Ft^3$. On a new set of axes, plot the four data points and this cubic. Compute the vectors $\boldsymbol{p}$ and $\boldsymbol{e}$, and the (squared) error $||\boldsymbol{e}||^2$.

2. In this problem we will prove that orthonormal vectors are linearly independent two different ways.

   (a) Vector proof: First, suppose that $c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + \cdots + c_k\boldsymbol{q}_k = \boldsymbol{0}$. Show that each $c_i = 0$. [*Hint*: Start by multipling both sides of the equation by $\boldsymbol{q}_i^T$.]

   (b) Matrix proof: Let $\boldsymbol{Q}$ be the matrix whose columns are the $\boldsymbol{q}_i$'s. Show that if $\boldsymbol{Qx} = \boldsymbol{0}$, then $\boldsymbol{x} = \boldsymbol{0}$. [*Hint*: Since $\boldsymbol{Q}$ need not be square, you cannot assume $\boldsymbol{Q}^{-1}$ exists, but $\boldsymbol{Q}^T$ of course will.]

3. For each of the following, answer either *true* (with a reason) or *false* (with a counterexample).

   (a) If $\boldsymbol{Q}$ is an orthogonal matrix, then $\boldsymbol{Q}^{-1}$ is orthogonal.

   (b) If $\boldsymbol{Q}$ is an orthogonal matrix, then $\boldsymbol{Q}^T$ is orthogonal.

   (c) If $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are orthogonal matrices, then $\boldsymbol{Q}_1\boldsymbol{Q}_2$ is orthogonal.

   (d) If $\boldsymbol{Q}$ is a matrix with orthonormal columns (need not be square), then $||\boldsymbol{Qx}|| = ||\boldsymbol{x}||$ for every $\boldsymbol{x}$.

4. What multiple of $\boldsymbol{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ should be subtracted from $\boldsymbol{b} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ to make the resulting vector $\boldsymbol{B}$ orthogonal to $\boldsymbol{a}$? Sketch a figure showing $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{B}$. Then normalize $\boldsymbol{A}$ and $\boldsymbol{B}$ to get an orthonormal set, $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

5. Let $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ be the (independent) column vectors of the matrix

$$\boldsymbol{M} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

   Use the Gram-Schmidt process to produce an orthonormal basis $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$. Then write $\boldsymbol{M} = \boldsymbol{QR}$, where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{R}$ is upper-triangular.

6. Recall that if $||\boldsymbol{u}|| = 1$, then the rank-1 matrix $\boldsymbol{uu}^T$ is the projection matrix onto $\boldsymbol{u}$. In this case, $\boldsymbol{Q} = \boldsymbol{I} - 2\boldsymbol{uu}^T$ is a *reflection matrix*.

   (a) Reflecting twice across the same axis is the identity. Verify that indeed, $\boldsymbol{Q}^2 = \boldsymbol{I}$.

   (b) Compute $\boldsymbol{Qu}$, and simplify this expression as much as possible.

   (c) Suppose $\boldsymbol{v}$ is orthogonal to $\boldsymbol{u}$. Compute $\boldsymbol{Qv}$, and simplify as much as possible.

   (d) Describe in plain English which subspace $\boldsymbol{Q}$ is reflecting across. Your answer should involve $\boldsymbol{u}$. Include a sketch.

   (e) Compute the reflection matrix $\boldsymbol{Q}_1 = \boldsymbol{I} - 2\boldsymbol{u}_1\boldsymbol{u}_1^T$ where $\boldsymbol{u}_1 = (0, 1)$. Compute $\boldsymbol{Q}_1\boldsymbol{x}_1$, where $\boldsymbol{x}_1 = (a, b)$, and sketch the vectors $\boldsymbol{u}_1$, $\boldsymbol{x}_1$, and $\boldsymbol{Q}_1\boldsymbol{x}_1$ in the plane.

   (f) Compute the reflection matrix $\boldsymbol{Q}_2 = \boldsymbol{I} - 2\boldsymbol{u}_2\boldsymbol{u}_2^T$ where $\boldsymbol{u}_2 = (0, \sqrt{2}/2, \sqrt{2}/2)$. Compute $\boldsymbol{Q}_2\boldsymbol{x}_2$, where $\boldsymbol{x}_2 = (1, 1, 1)$, and sketch the vectors $\boldsymbol{u}_2$, $\boldsymbol{x}_2$, and $\boldsymbol{Q}_2\boldsymbol{x}_2$ in $\mathbb{R}^3$.