Read: Strang, Section 5.1, 5.2, 5.3.

Suggested short conceptual exercises: Strang, Section 5.1, #1, 2, 4–6, 8, 11, 12, 17, 20, 28, 29. Section 5.2, #5–10, 23. Section 5.3, #4, 7, 9, 10, 14, 15, 21–23.

1. Use elemenatry row operations to compute the determinants of the following matrices.

$$\boldsymbol{A} = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

2. Recall that the determinant of a 2×2 matrix is $ad - bc$. Carry out the steps outlined below for the matrix $\boldsymbol{A}$. Then carry them out for $\boldsymbol{B}$.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}.$$

   (a) Compute the determinant of three matrices: $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ and $\boldsymbol{A} - \lambda \boldsymbol{I}$, where λ is a fixed parameter.

   (b) Determine which two numbers λ lead to $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$.

   (c) Write down the matrix $\boldsymbol{A} - \lambda \boldsymbol{I}$ for both of these values of λ. Note that these matrices should not be invertible.

3. Using linearity of each row, the determinant of an $n \times n$ matrix can be written as a sum of determinants of no more than $n!$ matrices that have *exactly one non-zero entry in each row and column*. For example, a 3×3 determinant breaks up as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \cdots.$$

   From here, it is easy to compute each individual determinant. Compute the determinant of each of the following matrices using this method. Only include the non-zero terms.

$$\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

   Use your answer to the first part to derive a "shortcut formula" for the determinant of any 3×3 matrix. [*Hint*: Write out the augmented 3×6 matrix $[\boldsymbol{A}|\boldsymbol{A}]$ and draw some "diagonal lines."]

4. Compute the determinants of the following matrices by *cofactor expansion*:

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 3 & 4 \\ 2 & 1 & 1 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 2 & -7 & 4 \\ 0 & 3 & 42 & 1 \\ 0 & 0 & 2 & 0 \\ -1 & 3 & 9 & -2 \end{bmatrix}.$$

To simplify your calculations, make a wise choice of which row or column to expand across.

5. The $n \times n$ determinant C_n has 1's above and below the main diagonal:

$$C_1 = |0|, \qquad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \qquad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}, \qquad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

   (a) Use cofactor expansions to compute the determinants C_1, C_2, C_3, and C_4.

   (b) Find the relation between C_n and C_{n-1} and C_{n-2}. Compute C_{10}.

6. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.

   (a) Find the cofactors of $\boldsymbol{A}$, put them into the cofactor matrix $\boldsymbol{C}$.

   (b) Use $\boldsymbol{A}$ and $\boldsymbol{C}$ to compute $\det \boldsymbol{A}$.

   (c) Use Part (b) to compute $\boldsymbol{A}^{-1}$.

   (d) Suppose that the 4 in $\boldsymbol{A}$ was changed to 100. Which of $\boldsymbol{C}$, $\det \boldsymbol{A}$, and $\boldsymbol{A}^{-1}$ would change?

7. Suppose $\boldsymbol{A}$ is an $n \times n$ matrix with integer entries.

   (a) Prove that if $\det \boldsymbol{A} = \pm 1$, then all entries of $\boldsymbol{A}^{-1}$ are integers.

   (b) Prove that if all entries of $\boldsymbol{A}^{-1}$ are integers, then $\det \boldsymbol{A} = \pm 1$.

8. The following matrix is called a (4×4) *Hadamard matrix*:

$$\boldsymbol{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

   Note that the "box" formed by the four row (or column) vectors is a hypercube in $\mathbb{R}^4$. Using this information alone, compute $|\det \boldsymbol{H}|$.