Read: Strang, Section 5.1, 5.2, 5.3.

Suggested short conceptual exercises: Strang, Section 5.1, #1, 2, 4–6, 8, 11, 12, 17, 20, 28, 29. Section 5.2, #5–10, 23. Section 5.3, #4, 7, 9, 10, 14, 15, 21–23.

1. Use elementary row operations to compute the determinants of the following matrices.

\[
A = \begin{bmatrix}
0 & a & 0 \\
0 & 0 & b \\
c & 0 & 0 \\
\end{bmatrix} \quad B = \begin{bmatrix}
0 & a & 0 \\
0 & 0 & b \\
0 & 0 & c \\
\end{bmatrix} \quad C = \begin{bmatrix}
a & a & a \\
a & b & b \\
a & b & c \\
\end{bmatrix}.
\]

2. Recall that the determinant of a 2 × 2 matrix is \(ad - bc\). Carry out the steps outlined below for the matrix \(A\). Then carry them out for \(B\).

\[
A = \begin{bmatrix}
2 & 1 \\
1 & 2 \\
\end{bmatrix} \quad B = \begin{bmatrix}
4 & 1 \\
2 & 3 \\
\end{bmatrix}.
\]

(a) Compute the determinant of three matrices: \(A\) and \(A^{-1}\) and \(A - \lambda I\), where \(\lambda\) is a fixed parameter.

(b) Determine which two numbers \(\lambda\) lead to \(\det(A - \lambda I) = 0\).

(c) Write down the matrix \(A - \lambda I\) for both of these values of \(\lambda\). Note that these matrices should not be invertible.

3. Using linearity of each row, the determinant of an \(n \times n\) matrix can be written as a sum of determinants of no more than \(n!\) matrices that have exactly one non-zero entry in each row and column. For example, a 3 × 3 determinant breaks up as

\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
\end{vmatrix} = a_{11} \begin{vmatrix} 0 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} + \cdots.
\]

From here, it is easy to compute each individual determinant. Compute the determinant of each of the following matrices using this method. Only include the non-zero terms.

\[
A = \begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{bmatrix} \quad B = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 2 & 0 & 1 \\
\end{bmatrix} \quad C = \begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 3 & 4 & 5 \\
5 & 4 & 0 & 3 \\
2 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

Use your answer to the first part to derive a “shortcut formula” for the determinant of any 3 × 3 matrix. \([\text{Hint: Write out the augmented 3} \times 6 \text{ matrix } [A|A] \text{ and draw some “diagonal lines.”}]\)

4. Compute the determinants of the following matrices by cofactor expansion:

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
1 & 3 & 4 \\
2 & 1 & 1 \\
\end{bmatrix} \quad B = \begin{bmatrix}
1 & 2 & 0 \\
0 & 3 & 0 \\
0 & 7 & 1 \\
\end{bmatrix} \quad C = \begin{bmatrix}
1 & 2 & -7 & 4 \\
0 & 3 & 4 & 1 \\
0 & 0 & 2 & 0 \\
-1 & 3 & 9 & -2 \\
\end{bmatrix}.
\]
To simplify your calculations, make a wise choice of which row or column to expand across.

5. The $n \times n$ determinant $C_n$ has 1’s above and below the main diagonal:

$$C_1 = \begin{vmatrix} 0 \end{vmatrix}, \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}, \quad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

(a) Use cofactor expansions to compute the determinants $C_1$, $C_2$, $C_3$, and $C_4$.

(b) Find the relation between $C_n$ and $C_{n-1}$ and $C_{n-2}$. Compute $C_{10}$.

6. Let $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.

(a) Find the cofactors of $A$, put them into the cofactor matrix $C$.

(b) Use $A$ and $C$ to compute $\det A$.

(c) Use Part (b) to compute $A^{-1}$.

(d) Suppose that the 4 in $A$ was changed to 100. Which of $C$, $\det A$, and $A^{-1}$ would change?

7. Suppose $A$ is an $n \times n$ matrix with integer entries.

(a) Prove that if $\det A = \pm 1$, then all entries of $A^{-1}$ are integers.

(b) Prove that if all entries of $A^{-1}$ are integers, then $\det A = \pm 1$.

8. The following matrix is called a $(4 \times 4)$ Hadamard matrix:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Note that the “box” formed by the four row (or column) vectors is a hypercube in $\mathbb{R}^4$. Using this information alone, compute $|\det H|$.