Read: Strang, Section 8.3, 8.5, 10.1, 10.2.

Suggested short conceptual exercises: Strang, Section 8.3, #1, 4, 6, 8–11, 17. Sect. 8.5, #5.

1. Consider the following permutation matrix: $\boldsymbol{P}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

   (a) Find the four (complex-valued) eigenvalues and eigenvectors of $\boldsymbol{P}_4$.

   (b) Write out the four eigenvalues you just found in polar form, $\lambda = Re^{i\theta}$. Write the corresponding eigenvectors in polar form, normalized so the first entry is 1, i.e., $\boldsymbol{v}_k = (e^{i0}, -, -, -) = (1, -, -, -)$.

   (c) Without actually computing them, venture a guess as to what the eigenvalues and eigenvectors of $\boldsymbol{P}_6$ are – the 6×6 matrix with five 1's below the diagonal and one in the upper-right corner.

2. Consider the following matrices

$$\boldsymbol{A}_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}, \qquad \boldsymbol{A}_2 = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}, \qquad \boldsymbol{A}_3 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

   (a) Diagonalize each matrix by writing $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$.

   (b) For each of these three matrices, compute the limit $\boldsymbol{A}^k = \boldsymbol{S}\boldsymbol{\Lambda}^k\boldsymbol{S}^{-1}$ as $k \to \infty$ if it exists.

   (c) Suppose $\boldsymbol{A}$ is an $n \times n$ matrix that is diagonalizable (so it has n linearly independent eigenvectors). What must be true for the limit $\boldsymbol{A}^k$ to exist as $k \to \infty$? What is needed for $\boldsymbol{A}^k \to \boldsymbol{0}$? Justify your answer.

   (d) Compute $(\boldsymbol{A}_3)^{10}\boldsymbol{u}_0$ for the following $\boldsymbol{u}_0$:

$$\boldsymbol{u}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad \boldsymbol{u}_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

3. Consider the following three Markov matrices.

$$\boldsymbol{A} = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}.$$

   Carry out the following steps for each matrix.

   (a) Draw a weighted directed graph on n (=2 or 3) vertices showing the transition probabilities between states.

   (b) Find the eigenvalues and steady-state eigenvector.

   (c) Diagonalize $\boldsymbol{A}$ by writing $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$ and compute $\lim_{k\to\infty} \boldsymbol{A}^k$.

4. Every year, 10% of young people become old, 0.1% of young people become dead, and 5% of old people become dead. Assume there are no births and no zombies.

   (a) Find the Markov matrix $\boldsymbol{A}$ that models this process:

   $$\begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k+1} = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k}.$$

   (b) Find the steady-state eigenvector by inspection alone.

   (c) Find the eigenvalues and eigenvectors of $\boldsymbol{A}$. (Use a computer.)

   (d) Suppose there are 1000 young people initially. Compute the number of young, old, and dead people after 10 years. (Again, use a computer, but also use $\boldsymbol{A}^k = \boldsymbol{S}\boldsymbol{\Lambda}^k\boldsymbol{S}^{-1}$.)

5. Consider the set of all complex-valued 2π-periodic functions, which means that $f(z+2\pi) = f(z)$ for all z. This is a vector space and the infinite set $\{e^{inx} : n \in \mathbb{Z}\}$ is a basis. Define an *inner product* (i.e., dot product) on this space by

   $$\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}\, dx.$$

   (a) Compute $\langle e^{inx}, e^{imx} \rangle$ and verify that this basis is indeed orthogonal. Recall that $\overline{e^{ix}} = e^{-ix}$, and be sure to consider the cases when $n = m$ and $n \neq m$ separately.

   (b) Since $\{e^{inx} : n \in \mathbb{Z}\}$ is a basis, we can write any 2π-periodic function $f(x)$ as

   $$f(x) = \quad \cdots + c_{-2}e^{-2ix} + c_{-1}e^{-ix} + c_0 + c_1e^{ix} + c_2e^{2ix} + \cdots \quad = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

   Derive a formula for c_n. [*Hint*: Right-multiply both sides of the above equation by $\overline{e^{inx}}$ and integrate.]

   (c) Give an *orthonormal basis* for this vector space.

   (d) Define the function $f(x) = e^x$ on the interval $[0, 2\pi]$, and extend f to be periodic. Sketch the graph of this function and compute its complex Fourier series. That is, compute the coefficients c_n. You'll need to compute c_n (for $n \neq 0$) and c_0 separately.

6. Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 2 & b \\ 1 & 0 \end{bmatrix}$.

   (a) What number b makes $\boldsymbol{A}^{-1}$ not exist?

   (b) What number b makes $\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$ impossible?

   (c) What number b in makes $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$ possible? Note that $\boldsymbol{Q}$ is *orthogonal*, since $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$.