Read: Strang, Section 8.3, 8.5, 10.1, 10.2.

Suggested short conceptual exercises: Strang, Section 8.3, #1, 4, 6, 8–11, 17. Sect. 8.5, #5.

1. Consider the following permutation matrix: \( P_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \)

   (a) Find the four (complex-valued) eigenvalues and eigenvectors of \( P_4 \).
   (b) Write out the four eigenvalues you just found in polar form, \( \lambda = Re^{i\theta} \). Write the corresponding eigenvectors in polar form, normalized so the first entry is 1, i.e., \( v_k = (e^{i\theta}, -, -, -) = (1, -, -, -) \).
   (c) Without actually computing them, venture a guess as to what the eigenvalues and eigenvectors of \( P_6 \) are – the \( 6 \times 6 \) matrix with five 1’s below the diagonal and one in the upper-right corner.

2. Consider the following matrices
   \( A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}, \quad A_3 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix} \).

   (a) Diagonalize each matrix by writing \( A = SAS^{-1} \).
   (b) For each of these three matrices, compute the limit \( A^k = SAS^{-1} \) as \( k \to \infty \) if it exists.
   (c) Suppose \( A \) is an \( n \times n \) matrix that is diagonalizable (so it has \( n \) linearly independent eigenvectors). What must be true for the limit \( A^k \) to exist as \( k \to \infty \)? What is needed for \( A^k \to 0 \)? Justify your answer.
   (d) Compute \( (A_3)^{10}u_0 \) for the following \( u_0 \):
      \[
      u_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad u_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad u_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.
      \]

3. Consider the following three Markov matrices.
   \[
   A = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}, \quad A = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}.
   \]

   Carry out the following steps for each matrix.
   (a) Draw a weighted directed graph on \( n \) (=2 or 3) vertices showing the transition probabilities between states.
   (b) Find the eigenvalues and steady-state eigenvector.
   (c) Diagonalize \( A \) by writing \( A = SAS^{-1} \) and compute \( \lim_{k \to \infty} A^k \).
4. Every year, 10% of young people become old, 0.1% of young people become dead, and 5% of old people become dead. Assume there are no births and no zombies.

(a) Find the Markov matrix $A$ that models this process:

$$
\begin{bmatrix}
\text{young} \\ \text{old} \\ \text{dead}
\end{bmatrix}_{k+1} = 
\begin{bmatrix}
- & - & - \\ - & - & - \\ - & - & -
\end{bmatrix}
\begin{bmatrix}
\text{young} \\ \text{old} \\ \text{dead}
\end{bmatrix}_k.
$$

(b) Find the steady-state eigenvector by inspection alone.

(c) Find the eigenvalues and eigenvectors of $A$. (Use a computer.)

(d) Suppose there are 1000 young people initially. Compute the number of young, old, and dead people after 10 years. (Again, use a computer, but also use $A^k = S\Lambda^k S^{-1}$.)

5. Consider the set of all complex-valued $2\pi$-periodic functions, which means that $f(z+2\pi) = f(z)$ for all $z$. This is a vector space and the infinite set $\{e^{inx}: n \in \mathbb{Z}\}$ is a basis. Define an *inner product* (i.e., dot product) on this space by

$$
\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)} \, dx.
$$

(a) Compute $\langle e^{inx}, e^{imx} \rangle$ and verify that this basis is indeed orthogonal. Recall that $e^{ix} = e^{-ix}$, and be sure to consider the cases when $n = m$ and $n \neq m$ separately.

(b) Since $\{e^{inx}: n \in \mathbb{Z}\}$ is a basis, we can write any $2\pi$-periodic function $f(x)$ as

$$
f(x) = \cdots + c_{-2}e^{-2ix} + c_{-1}e^{-ix} + c_0 + c_1e^{ix} + c_2e^{2ix} + \cdots = \sum_{n=-\infty}^{\infty} c_n e^{inx}.
$$

Derive a formula for $c_n$. [*Hint: Right-multiply both sides of the above equation by $\overline{e^{inx}}$ and integrate.*]

(c) Give an *orthonormal basis* for this vector space.

(d) Define the function $f(x) = e^x$ on the interval $[0, 2\pi]$, and extend $f$ to be periodic. Sketch the graph of this function and compute its complex Fourier series. That is, compute the coefficients $c_n$. You’ll need to compute $c_n$ (for $n \neq 0$) and $c_0$ separately.

6. Consider the matrix $A = \begin{bmatrix} 2 & b \\ 1 & 0 \end{bmatrix}$.

(a) What number $b$ makes $A^{-1}$ not exist?

(b) What number $b$ makes $A = S\Lambda S^{-1}$ impossible?

(c) What number $b$ in makes $A = Q\Lambda Q^T$ possible? Note that $Q$ is *orthogonal*, since $Q^{-1} = Q^T$. 

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