Read: Strang, Section 6.4, 6.5.

Suggested short conceptual exercises: Strang, Section 6.4, #2, 7–10, 13–15, 18, 20–22, 25, 26. Section 6.5, #1, 6, 10, 14–20, 27–31, 33.

1. Diagonalize the following matrices into $\boldsymbol{Q\Lambda Q}^T$, where $\boldsymbol{Q}$ is an *orthogonal* matrix.

$$\boldsymbol{A} = \begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & -2 & 2 \\ -2 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

   Find *all* orthogonal matrices that diagonalize $\boldsymbol{A}$. How many will diagonalize $\boldsymbol{B}$?

2. Which of these classes of matrices do $\boldsymbol{A}$ and $\boldsymbol{B}$ belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{B} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

   Which of these factorizations are possible for $\boldsymbol{A}$ and $\boldsymbol{B}$: $\boldsymbol{LU}$, $\boldsymbol{QR}$, $\boldsymbol{S\Lambda S}^{-1}$, $\boldsymbol{Q\Lambda Q}^T$?

3. Let $\boldsymbol{A}$ be an $n\times n$ *Hermitian* matrix (also called *self-adjoint*), which means that $\overline{\boldsymbol{A}}^T = \boldsymbol{A}$. Note that $\boldsymbol{A}$ may have complex-valued entries.

   (a) Which Hermitian matrices are also symmetric?

   (b) Prove that all eigenvalues of $\boldsymbol{A}$ are real.

   (c) Prove that if $\boldsymbol{Av} = \lambda\boldsymbol{v}$ and $\boldsymbol{Aw} = \mu\boldsymbol{w}$ for $\lambda \neq \mu$, then $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal.

   These two proofs can be done by modifying the arguments in class that proved these statements for real symmetric matrices.

4. An real-valued $n \times n$ matrix is *skew-symmetric* if $\boldsymbol{A}^T = -\boldsymbol{A}$.

   (a) What must be true about the diagonal entries of a skew-symmetric matrix?

   (b) Prove that every non-zero eigenvalue of a skew-symmetric matrix is purely imaginary. [*Hint*: Note that a complex eigenvalue $\lambda = a+bi$ is real if $\lambda = \overline{\lambda}$, whereas it is purely imaginary if $\lambda = -\overline{\lambda}$.]

   (c) Prove that if n is odd and $\boldsymbol{A}$ is skew-symmetric, then $\boldsymbol{A}$ must be singular. [*Hint*: complex eigenvalues come in pairs, because $\boldsymbol{Av} = \lambda\boldsymbol{v}$ implies $\overline{\boldsymbol{Av}} = \boldsymbol{A}\overline{\boldsymbol{v}} = \overline{\lambda}\overline{\boldsymbol{v}}$.]

   (d) Write $\boldsymbol{A} = \boldsymbol{M} + \boldsymbol{N}$, the sum of a *symmetric* and a *skew-symmetric* matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 0 \\ 7 & 6 & 5 \end{bmatrix} = \boldsymbol{M} + \boldsymbol{N} \qquad (\boldsymbol{M}^T = \boldsymbol{M},\ \ \boldsymbol{N}^T = -\boldsymbol{N}).$$

   Hint: For any square matrix, $\boldsymbol{M} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$. What does this force $\boldsymbol{N}$ to be?

5. Consider the following two quadratic forms:

$$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 5x_2^2, \qquad\qquad f(x_1, x_2) = 2x_1^2 + 6x_1x_2 + 2x_2^2.$$

Carry out the following steps for each of these functions.

(a) Write $f(x_1, x_2) = \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ for some $\boldsymbol{A}$.

(b) Diagonalize $\boldsymbol{A}$ by writing $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$ for an orthogonal matrix $\boldsymbol{Q}$ that additionally has $\det \boldsymbol{Q} = +1$. [*Hint*: Every 2×2 orthogonal matrix with determinant 1 is a *rotation matrix* $\boldsymbol{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for some $0 \leq \theta < 2\pi$. Find θ.]

(c) Introduce new coordinates by setting $\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{y}$, and substitute $\boldsymbol{y} = \boldsymbol{Q}^T\boldsymbol{x}$ into the quadratic form $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$. Write the result both as an equation $f(y_1, y_2)$, and in matrix form, $\boldsymbol{y}^T\boldsymbol{\Lambda}\boldsymbol{y}$.

(d) Sketch the conic section $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 1$ on the x_1x_2-plane, and sketch $\boldsymbol{y}^T\boldsymbol{\Lambda}\boldsymbol{y} = 1$ on the y_1y_2-plane (okay to use a computer).

(e) Write $f(x_1, x_2)$ as either the sum or the difference of two squares. (Only one will be possible.)

6. Consider the following quadratic form:

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & \boldsymbol{A} & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

Find the 3×3 matrix $\boldsymbol{A}$ and its pivots, rank, eigenvalues, and three (upper-left) subdeterminants: