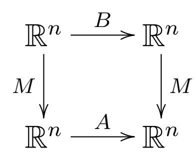
Read: Strang, Section 6.6, 6.7.

Suggested short conceptual exercises: Strang, Section 6.6, #1, 4, 7, 8, 12–15, 17, 20. Strang, Section 6.7, #3, 9–13, 15.

1. Suppose $\boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$ is an $n \times n$ matrix. The relationship between $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{M}$ as functions $\mathbb{R}^n \to \mathbb{R}^n$ is shown in the following *commutative diagram*:



   Remember that matrix multiplication represents function composition, and so should be read from *right-to-left*.

   (a) Draw a commutative diagram showing $\boldsymbol{B}^2 = (\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M})(\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}) = \boldsymbol{M}^{-1}\boldsymbol{A}^2\boldsymbol{M}$. [*Hint*: Imagine "stacking" two diagrams horizontally.]

   (b) Suppose that $\boldsymbol{B} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$ and $\boldsymbol{C} = \boldsymbol{N}^{-1}\boldsymbol{B}\boldsymbol{N}$. Draw a commutative diagram showing how $\boldsymbol{A}$ is similar to $\boldsymbol{C}$. Write this out algebraically as well. You have just proven that similarity is *transitive*.

   (c) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvalues $\lambda_1, \dots, \lambda_n$, all distinct. Prove that $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar.

   (d) Show by example how the result in Part (c) fails if the eigenvalues are not distinct.

2. Show that each pair $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ are similar by finding $\boldsymbol{M}_i$ such that $\boldsymbol{B}_i = \boldsymbol{M}_i^{-1}\boldsymbol{A}_i\boldsymbol{M}_i$.

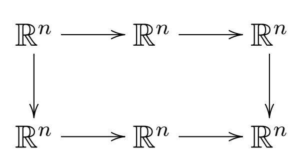
$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ \boldsymbol{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{A}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ \boldsymbol{B}_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \ \boldsymbol{A}_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ \boldsymbol{B}_3 = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.$$

3. There are sixteen 2×2 matrices whose entries are 0's and 1's. Partition these matrices into "families" (equivalence classes) where similar matrices go into the same family.

4. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices with $\boldsymbol{B}$ invertible.

   (a) Prove that $\boldsymbol{A}\boldsymbol{B}$ is similar to $\boldsymbol{B}\boldsymbol{A}$.

   (b) Illustrate your proof from Part (a) by correctly labeling the six maps in the following commutative diagram:



   (c) Conclude that $\boldsymbol{A}\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}$ have the same eigenvalues.

5. Consider the following Jordan blocks with eigenvalue λ:

$$\boldsymbol{J}_2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad \boldsymbol{J}_3 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \qquad \boldsymbol{J}_4 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

For each $\boldsymbol{J}_i$, compute $\boldsymbol{J}_i^2$ and $\boldsymbol{J}_i^3$. Guess the form of $\boldsymbol{J}_i^k$. Set $k = 0$ to find $\boldsymbol{J}_i^0$ and $k = -1$ to find $\boldsymbol{J}_i^{-1}$.

6. Suppose $\boldsymbol{A}$ is a 4×4 matrix that has exactly two distinct eigenvalues, $\lambda = 0$ and $\lambda = 2$, but you do not know how many of each occurs.

   (a) Write down a list of matrices such that $\boldsymbol{A}$ must is similar to exactly one matrix on your list.

   (b) For each matrix above, find the number of linearly independent eigenvectors for $\lambda = 0$ and for $\lambda = 2$. Recall that this is the dimension of $N(A - \lambda I)$.

7. Consider the following matrices: $\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $\boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$, $\boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$.

   (a) Find the eigenvalues σ_1^2, σ_2^2 and unit eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ of $\boldsymbol{A}^T\boldsymbol{A}$.

   (b) For the $\sigma_i \neq 0$, compute $\boldsymbol{u}_i = \boldsymbol{A}\boldsymbol{v}_i/\sigma_i$ and verify that indeed $||\boldsymbol{u}_i|| = 1$. Find the other $\boldsymbol{u}_i$ by computing the other unit eigenvector of $\boldsymbol{A}\boldsymbol{A}^T$.

   (c) Write out the singular value decomposition (SVD), $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$.

   (d) Write down orthonormal bases for the four fundamental subspaces of $\boldsymbol{A}$.

   (e) Describe *all* matrices that have the same four fundamental subspaces.