1. Suppose \( B = M^{-1}AM \) is an \( n \times n \) matrix. The relationship between \( A, B, \) and \( M \) as functions \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) is shown in the following commutative diagram:

Remember that matrix multiplication represents function composition, and so should be read from right-to-left.

(a) Draw a commutative diagram showing \( B^2 = (M^{-1}AM)(M^{-1}AM) = M^{-1}A^2M \). [Hint: Imagine “stacking” two diagrams horizontally.]

(b) Suppose that \( B = M^{-1}AM \) and \( C = N^{-1}BN \). Draw a commutative diagram showing how \( A \) is similar to \( C \). Write this out algebraically as well. You have just proven that similarity is transitive.

(c) Suppose \( A \) and \( B \) have the same eigenvalues \( \lambda_1, \ldots, \lambda_n \), all distinct. Prove that \( A \) and \( B \) are similar.

(d) Show by example how the result in Part (c) fails if the eigenvalues are not distinct.

2. Show that each pair \( A_i \) and \( B_i \) are similar by finding \( M_i \) such that \( B_i = M_i^{-1}A_iM_i \).

\[
A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.
\]

3. There are sixteen \( 2 \times 2 \) matrices whose entries are 0’s and 1’s. Partition these matrices into “families” (equivalence classes) where similar matrices go into the same family.

4. Let \( A \) and \( B \) be \( n \times n \) matrices with \( B \) invertible.

(a) Prove that \( AB \) is similar to \( BA \).

(b) Illustrate your proof from Part (a) by correctly labeling the six maps in the following commutative diagram:

(c) Conclude that \( AB \) and \( BA \) have the same eigenvalues.
5. Consider the following Jordan blocks with eigenvalue $\lambda$:

\[ J_2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad J_3 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \quad J_4 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}. \]

For each $J_i$, compute $J_i^2$ and $J_i^3$. Guess the form of $J_i^k$. Set $k = 0$ to find $J_i^0$ and $k = -1$ to find $J_i^{-1}$.

6. Suppose $A$ is a $4 \times 4$ matrix that has exactly two distinct eigenvalues, $\lambda = 0$ and $\lambda = 2$, but you do not know how many of each occurs.

(a) Write down a list of matrices such that $A$ must is similar to exactly one matrix on your list.

(b) For each matrix above, find the number of linearly independent eigenvectors for $\lambda = 0$ and for $\lambda = 2$. Recall that this is the dimension of $N(A - \lambda I)$.

7. Consider the following matrices: $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $A^T A = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$, $A A^T = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$.

(a) Find the eigenvalues $\sigma_1^2$, $\sigma_2^2$ and unit eigenvectors $v_1$, $v_2$ of $A^T A$.

(b) For the $\sigma_i \neq 0$, compute $u_i = A v_i / \sigma_i$ and verify that indeed $||u_i|| = 1$. Find the other $u_i$ by computing the other unit eigenvector of $A A^T$.

(c) Write out the singular value decomposition (SVD), $A = U \Sigma V^T$.

(d) Write down orthonormal bases for the four fundamental subspaces of $A$.

(e) Describe all matrices that have the same four fundamental subspaces.