1 Geometry of Linear equations

Read Section 1.1, 1.2, 2.1

* Fundamental problem of linear algebra:

Solve n linear equations in n variables.

3 ways to think about this problem.

1. Row picture (you've seen this)
2. Column picture (probably new)
   - 2' Linear map/grid picture (new)
3. Matrix form (algebraic approach)

Example:

$$\boxed{\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}}$$

Let's do 3, the matrix form first.

Write

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

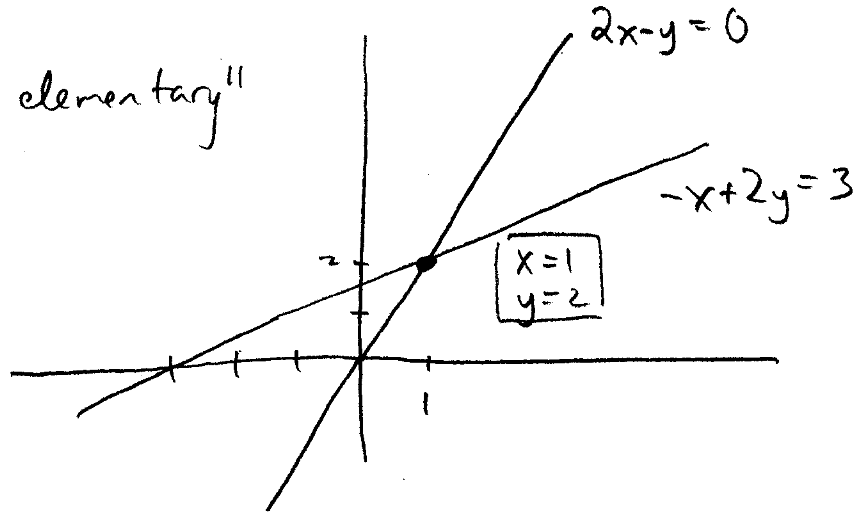
$$A \quad \vec{x} = \vec{b}$$

Goals:
- learn how to solve them
- Understand the bigger picture.

2

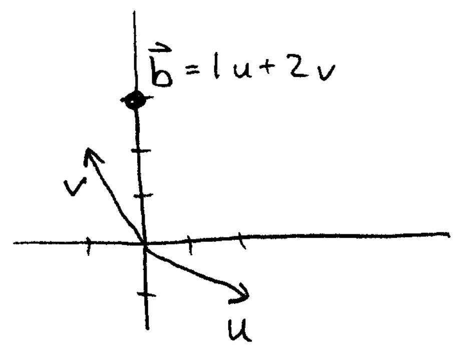
① Row picture: "most elementary"

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$



② Column picture: $\underbrace{x\begin{bmatrix}2\\-1\end{bmatrix} + y\begin{bmatrix}-1\\2\end{bmatrix}}_{\text{"linear combination" of the column vectors.}} = \begin{bmatrix}0\\3\end{bmatrix}$

Solution: $1\begin{bmatrix}2\\-1\end{bmatrix} + 2\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$



Question: What is the collection of all linear combinations of $\begin{bmatrix}2\\-1\end{bmatrix}$ and $\begin{bmatrix}-1\\2\end{bmatrix}$? (We call this the set spanned by $\left\{\begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\}$

Ans: We'd get the entire xy-plane. (Why?)

②' Linear map / grid picture.

Consider the matrix $A = \begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}$ as a linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

$$\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x - y\\-x + 2y\end{bmatrix}$$

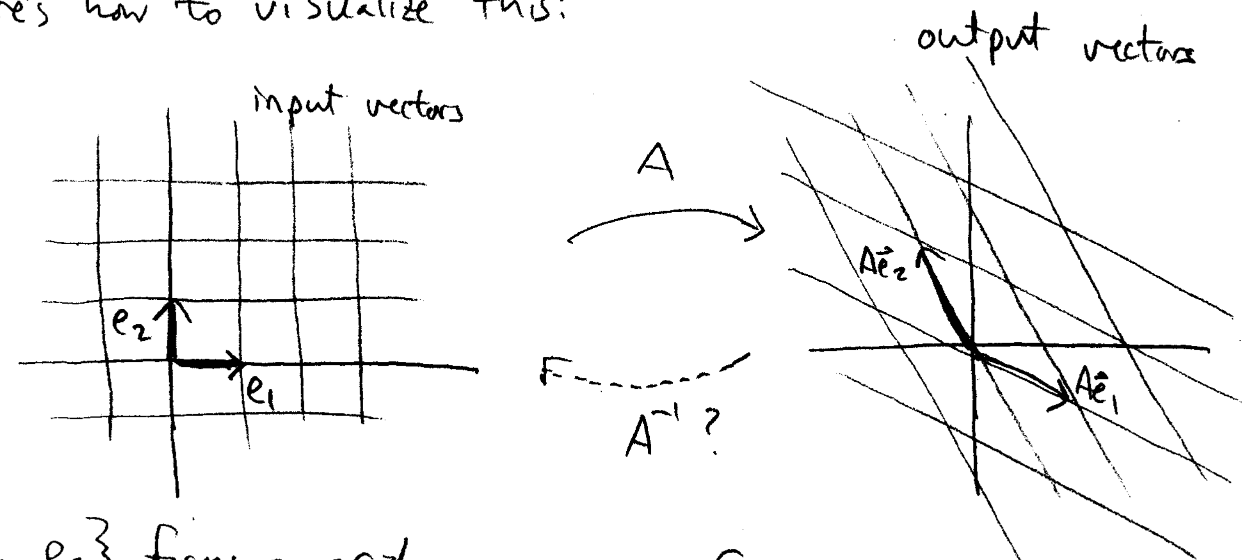
input vector ↗ ↖ output vector

Question: Where do the standard unit basis vectors,

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ get mapped to?

Ans: $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{e}_1} = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\text{column } 1}$ $\qquad$ $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{e}_2} = \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\text{column } 2}$

Here's how to visualize this:

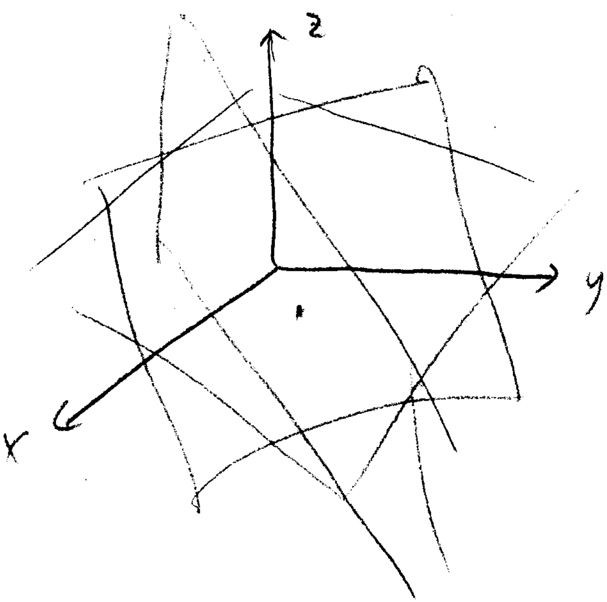


$\{e_1, e_2\}$ forms a grid $\qquad$ $\{Ae_1, Ae_2\}$ forms a grid.

Consider a 3x3 example.

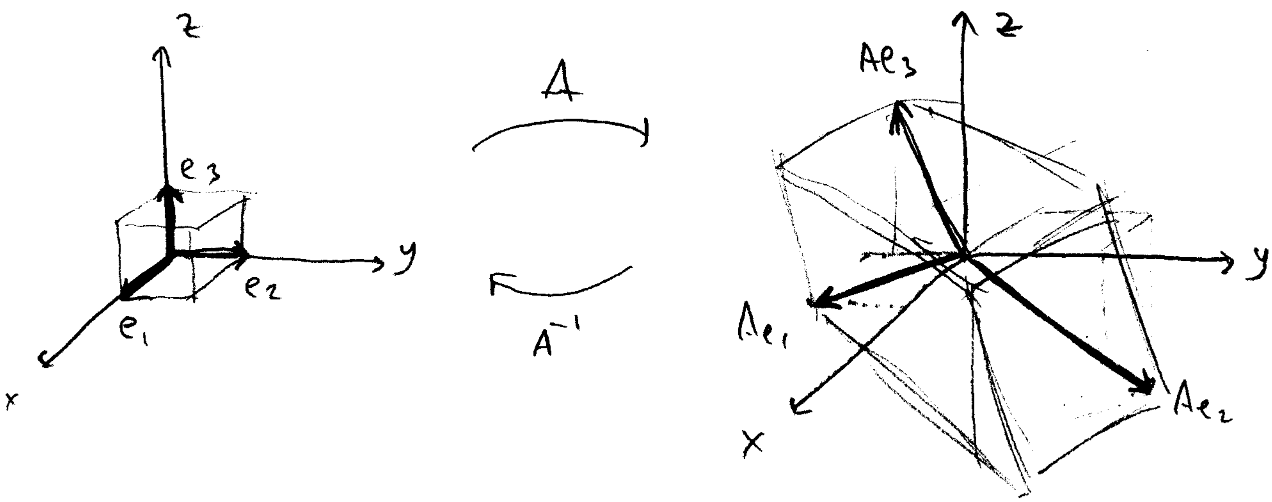
$$\begin{aligned} 2x - y \quad\;\; &= 0 \\ -x + 2y - z &= -1 \\ -3y + 4z &= 4 \end{aligned}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$



Moral: Row picture isn't nice in more than 2 dimensions.

Column picture: $x\begin{bmatrix}2\\-1\\0\end{bmatrix} + y\begin{bmatrix}-1\\2\\-3\end{bmatrix} + z\begin{bmatrix}0\\-1\\4\end{bmatrix} = \begin{bmatrix}0\\-1\\4\end{bmatrix}$



Sol'n: $\underbrace{x=0,\ y=0,\ z=1}$

The $(0,0,1)$ point on the "column space grid."

Another advantage of the "column picture"

Consider a similar equation (same A, different $\vec{b}$).

$x\begin{bmatrix}2\\-1\\0\end{bmatrix} + y\begin{bmatrix}-1\\2\\-3\end{bmatrix} + z\begin{bmatrix}0\\-1\\4\end{bmatrix} = \begin{bmatrix}1\\1\\-3\end{bmatrix}$ sol'n: $x=1,\ y=1,\ z=0.$

Row picture: 3 different planes!

Column picture: Same "grid," just a different point.

<u>Question</u>: Can we solve $A\vec{x} = \vec{b}$ for every $\vec{b}$?

i.e., Do the <u>linear combinations</u> of the columns "fill up" 3-dimensional space?

<u>Ans</u>: For this matrix: yes

In general: no.

<u>what could go wrong</u>: What if all 3 column vectors lie in the same plane?

Then we're "stuck" in that plane.

Recall <u>matrix multiplication</u>:

By <u>rows</u>:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

By <u>columns</u>

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

In general: