$\boxed{3}$ Elimination with matrices

Read Section 2.2, 2.3.

Example: Solve

$$\begin{aligned} x+2y+z&=2\\ 3x+8y+z&=12\\ 4y+z&=2 \end{aligned}
\qquad
\overset{A}{\begin{bmatrix}1&2&1\\3&8&1\\0&4&1\end{bmatrix}}\overset{\vec{x}}{\begin{bmatrix}x\\y\\z\end{bmatrix}}\overset{=}{=}\overset{\vec{b}}{\begin{bmatrix}2\\12\\2\end{bmatrix}}$$

1st pivot

$$\left[\begin{array}{ccc|c}\boxed{1}&2&1&2\\3&8&1&12\\0&4&1&2\end{array}\right]
\xrightarrow[R2-3\cdot R1]{E_{21}}
\left[\begin{array}{ccc|c}\boxed{1}&2&1&2\\0&\boxed{2}&-2&6\\0&4&1&2\end{array}\right]
\xrightarrow[R3-2\cdot R2]{E_{32}}
\left[\begin{array}{ccc|c}\boxed{1}&2&0&2\\0&\boxed{2}&-2&6\\0&0&\boxed{5}&-10\end{array}\right]$$

Call this U "upper triangular" $\qquad \vec{c}$

Question: How could this fail?

What if the 1st pivot was 0? (Swap rows)

What if the 2nd pivot was 0? (swap rows)

What if the 3rd pivot was 0? (out of luck)

We've turned $A\vec{x}=\vec{b}$ into $U\vec{x}=\vec{c}$.

$$\begin{aligned} x+2y+z&=2\\ 2y-2z&=6\\ 5z&=-10 \end{aligned}
\quad \overset{\text{"back substitution"}}{\Longrightarrow} \quad
\begin{aligned} x&=2\\ y&=1\\ z&=-2 \end{aligned}
\quad \Longrightarrow \quad \vec{x}=\begin{bmatrix}2\\1\\-2\end{bmatrix}$$

<u>Remark</u>: Instead of back-substituting, we can solve the system by eliminating the entries above the diagonal.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right) \xrightarrow[\text{rows}]{\text{scale}} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow{E_{23}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{E_{13}} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow{E_{12}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right) \Rightarrow \vec{x} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

<u>Matrices</u>: Columns vs. rows.

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 3\cdot(\text{col } 1) + 4(\text{col } 2) + 5(\text{col } 3)$$

$$\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = 3\cdot(\text{row } 1) + 4(\text{row } 2) + 5(\text{row } 3).$$

<u>Example</u>: Subtract $3\cdot(\text{Row } 1)$ from Row 2. (This was <u>Step 1</u>)

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

E_{21} (made (2,1)-entry 0).

Step 2: Subtract 2·(Row 2) from Row 3.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{32}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}}_{U}$$

matrix multiplication is associative!

Summary: $E_{32}(E_{21}A) = U$, or $(E_{32}E_{21})A = U$

We call E_{ij} an elementary matrix.

There are other types of elementary matrices:

- Row multiplication:

Ex: $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

- Permutation matrices (exchanges two rows)

Ex: $$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_{12}} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 8 & 1 \\ 1 & 2 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

What about exchanging columns? Need to right-multiply.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 8 & 3 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

Summary: Column-operations: Right-multiply

Row-operations: Left-multiply.

Inverses of elementary matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Add 3 (Row 1) to Row 2

Subtract 3·(Row 1) from Row 2