5 LU-Factorization

First, we need to establish some basic facts

Fact: $\boxed{(AB)^T = B^T A^T}$

Consider the equation $AA^{-1} = I$

Apply T: $(AA^{-1})^T = I^T$

$$(A^{-1})^T A^T = I \Rightarrow \boxed{(A^T)^{-1} = (A^{-1})^T}$$

2×2 Example: $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$\Rightarrow \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

lower-triangular diagonal upper-triangular

3×3 Example: (Assume no row-exchanges)

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \qquad E_{32}E_{31}E_{21}A = U \Rightarrow A = \underbrace{\boxed{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}}_{L} U$$

Why do we prefer using <u>inverses</u> of elementary matrices.

<u>Ex:</u>

$$\overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}}\overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} = E \qquad EA = U$$

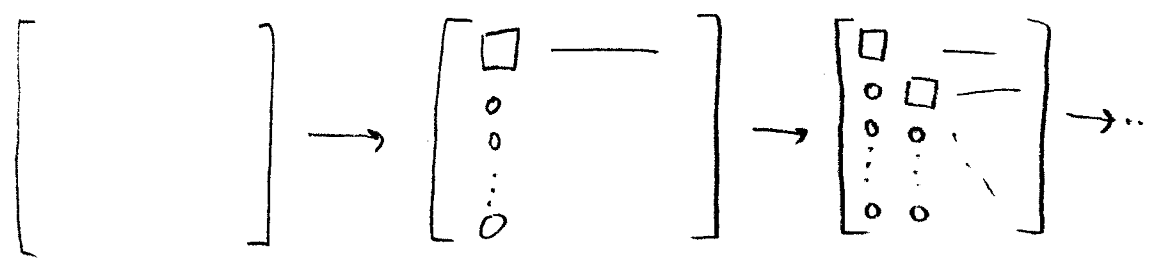
vs.

$$\overset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\overset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L \qquad A = LU$$

<u>Moral</u>: L keeps a record of where those multipliers were.

<u>Question:</u> How many operations does it take to reduce an $n \times n$ matrix to upper-triangular form. ($n, n^2, n^3, 2^n, n!$, etc. ?)

<u>Ex:</u> $n = 100$



$\approx 100^2$ steps $\qquad \approx 99^2$ steps

Total # of steps $\approx n^2 + (n-1)^2 + \cdots + 2^2 + 1^2 \approx \boxed{\frac{1}{3}n^3}$ (looks like $\int x^2 dx$)

Cost for reducing $A \approx \frac{1}{3}n^3$

Cost for reducing $\vec{b} \approx n^2$.

Permutation matrices

3x3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \overset{P_{12}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}, \quad \overset{P_{13}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}, \quad \overset{P_{23}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ "3-cycles"

What if we multiply 2 of these?

What if we take inverses.

Take MthSc 412.

This set is "closed" under these operations. It is a group!

Remark: $P^{-1} = P^T$ (Why?)

Recall that A^T is the transpose of A, defined by

$$(A^T)_{ij} = A_{ji}.$$

Permutations execute row exchanges (when left-multiplied).

So if we need these $A = LU$ factorization becomes $\boxed{PA = LU}$

Note: There are $n! = n(n-1)\cdots 3\cdot 2\cdot 1$ permutation matrices of size n.

Symmetric matrices: $\boxed{A^T = A}$ Ex: $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$.

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Fact: A^TA is always symmetric.

Ex: $\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$

Why? Take transpose: $(A^TA)^T = A^T(A^T)^T = A^TA.$