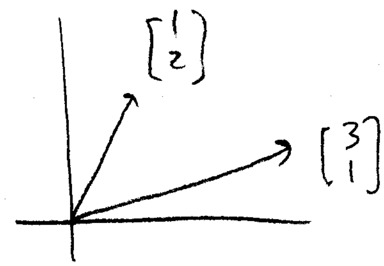
$\boxed{6}$ Vector spaces, Column space, & Row space

Basic operations: $\underbrace{\text{Addition, subtraction, scalar multiplication}}_{\text{"linear combinations"}}$

This means that if v, w are in the space, then

$v+w$ and cv are in the space too (c = any scalar).

Examples: $\mathbb{R}^2$ = all 2-dim'l real vectors, or "xy-plane"

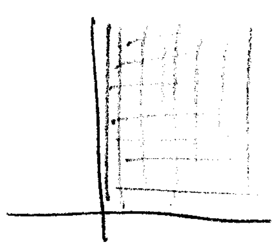


$\mathbb{R}^3$ = all (column) vectors with 3 components.

$\mathbb{R}^n$ = all column vectors with n components.

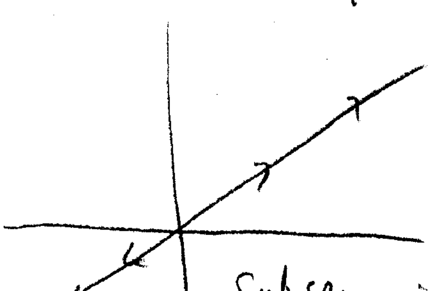
Remark: We also need obvious rules for $+$ and $\cdot$ (distributive & associative laws, etc)

Non-example:



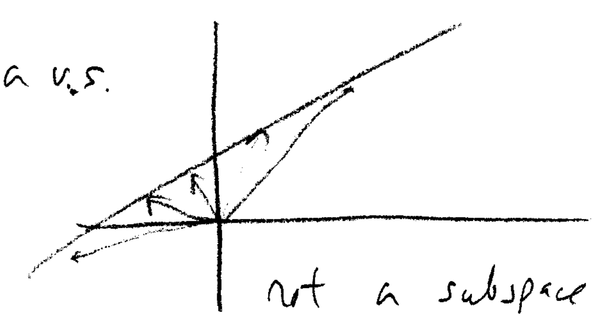
Not closed under scalar mult. e.g., $-3\begin{bmatrix}2\\1\end{bmatrix}$.

v.s.



Subspace of $\mathbb{R}^2$

not a v.s.



not a subspace

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Remark: Every vector space (and subspace) must contain $\vec{0}$.

Subspaces of $\mathbb{R}^2$:

(2) $\mathbb{R}^2$

(1) Any line through $\vec{0} := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(0) $\vec{0}$-vector only

Subspaces of $\mathbb{R}^3$:

(3) $\mathbb{R}^3$

(2) Plane through $\vec{0} := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

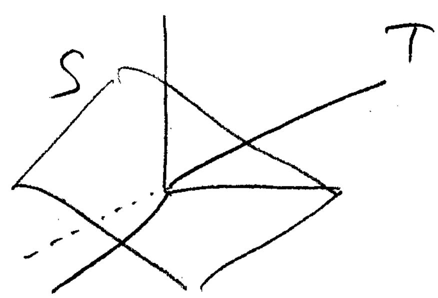
(1) Line through $\vec{0}$.

(0) $\vec{0}$-vector only.

Let S, T be subspaces.

Question: Is $S \cup T :=$ "all vectors in S or T" a subspace?

Ans: No. (Why?)



Question: Is $S \cap T :=$ "all vectors in S and T" a subspace?

Ans: Yes. (Why?)

Subspaces from matrices

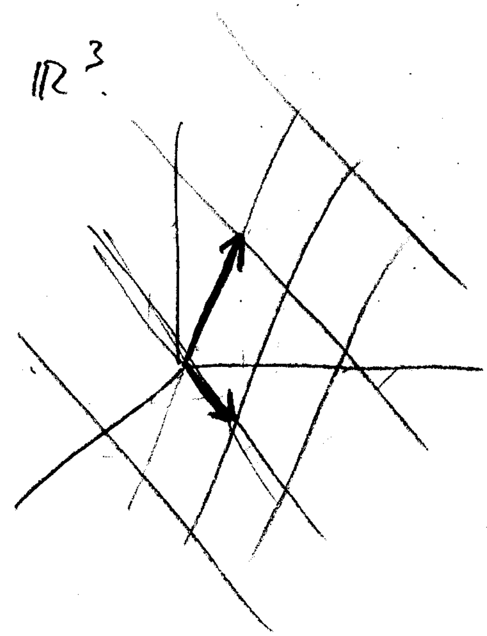
$\underline{\text{Column space of } A} = \overbrace{\text{set of all linear combinations}}^{\text{the "span" of the column vectors}}$ of the column vectors of A, denoted $C(A)$.

Ex 1: $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$. Columns are in $\mathbb{R}^3$.

$C(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \right\}$

This is a plane in $\mathbb{R}^3$

(Think: This is the "grid"!)



What would this picture be like for a 10×5 matrix?

Ex 2: $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$

Column space $C(A)$ is a (2-dim'l subspace of $\mathbb{R}^4$).

Question: For which $\vec{b}$ does $A\vec{x} = \vec{b}$ have a sol'n?

$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ Try $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

✓ ✓ ✓

Ans: Exactly when $\vec{b}$ is in $C(A)$!

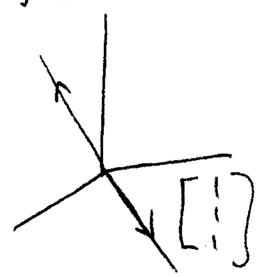
Nullspace of A = All solutions $\vec{x}$ to $\boxed{A\vec{x}=\vec{0}}$, denoted $N(A)$.

Ex 2: $$A\vec{x} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Remark: $C(A) \subseteq \mathbb{R}^4$ but $N(A) \subseteq \mathbb{R}^3$.

Null space $N(A)$ contains: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} c \\ c \\ -c \end{bmatrix}$.

It is a line in $\mathbb{R}^3$.



Remark: Solutions to $Ax=0$ always give a subspace.

Check: If $Av=0$ and $Aw=0$, then $A(v+w) = Av + Aw = 0$ ✓

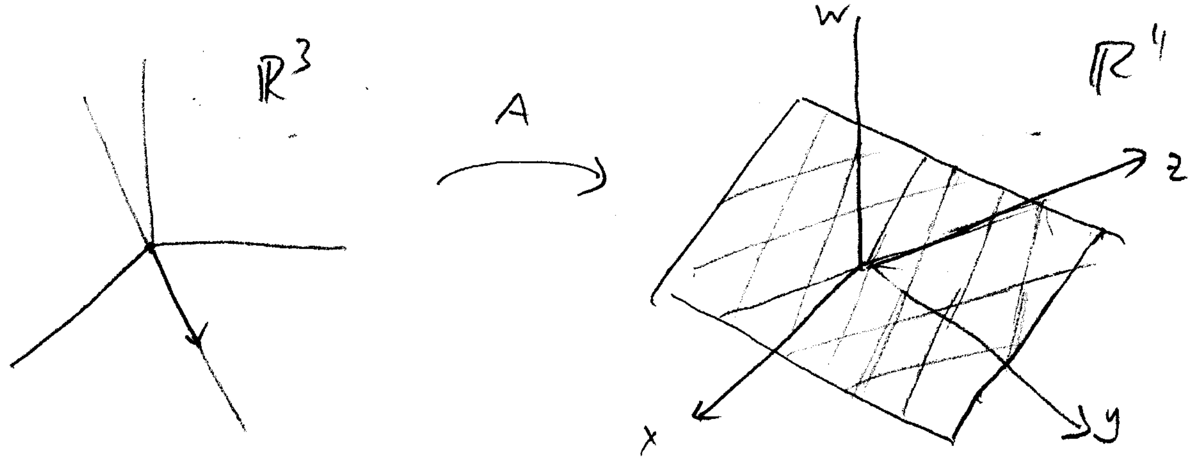
Also, $A(cv) = c\,Av = c \cdot 0 = 0$. ✓

Note: We could have done just $A(cv+dw) = c\,Av + d\,Aw = 0$.

The grid picture:

The matrix A is a linear map $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$



$N(A)$ = set of all vectors that get "killed" by A

$C(A)$ = Range(A) = set of a vectors that get hit over here.