7 Solving $Ax = 0$: Pivot & free variables

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

Note: Col. 2 & row 3 are not independent.

Algorithm: Elimination, extended to rectangular matrices

Remark: Elimination does not change the null space.

But it changes the column space.

"echelon form"

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & \boxed{0} & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

tells us col 2 is dependent

move onto this pivot.

pivot columns

free columns

Def: The rank of A = # of pivots (here, $r = 2$).

Next step: Assign anything we want to free variables x_2 & x_4.

$$\begin{cases} x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \\ 2x_3 + 2x_4 = 0 \end{cases}$$

$$Ux = 0$$

Ex: $x_2 = 1$, $x_4 = 0$ $\Rightarrow$ $x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

"Special solutions"

$x_2 = 0$, $x_4 = 1$ $\Rightarrow$ $x = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

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<u>All solutions</u>: $x = c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$ $r = 2$ pivot vars.
$n - r = 2$ free vars.

we can continue with elimination: Notice $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ in pivot rows/cols.

$$\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix} \longrightarrow \begin{bmatrix}\boxed{1}&2&0&-2\\0&0&\boxed{1}&2\\0&0&0&0\end{bmatrix} = R.$$

<u>R</u>educed <u>r</u>ow <u>e</u>chelon <u>f</u>orm: zeros above & below pivots
all pivots scaled to be 1.

$$\begin{cases} x_1 + 2x_2 \quad - 2x_4 = 0 \\ \qquad x_3 + 2x_4 = 0 \end{cases}$$

$Rx = 0$

$\begin{bmatrix}1&0\\0&1\end{bmatrix} \leftarrow I$ pivot cols $\begin{bmatrix}2&-2\\0&2\end{bmatrix} \leftarrow F$ free cols.

<u>Note</u>: Sol'ns to $Ax = 0$, $Ux = 0$, $Rx = 0$ all the same.

rref form: $R = \begin{bmatrix}I & F\\0&0\end{bmatrix} \leftarrow r$ pivot rows

r pivot cols $n-r$ free cols.

<u>Want</u>: "Nullspace matrix" N solving $RN = 0$ columns are the "special solutions"

Note that $\begin{bmatrix}I & F\\0&0\end{bmatrix}\begin{bmatrix}-F\\I\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$Rx = 0$ is $[I \;\; F]\begin{bmatrix}x_{pivot}\\x_{free}\end{bmatrix} = 0$ $\Rightarrow I x_{pivot} + F x_{free} = 0$

$\Rightarrow x_{pivot} = -F x_{free}$

Ex 2: $A^T = \begin{bmatrix}1&2&3\\2&4&6\\2&6&8\\2&8&10\end{bmatrix} \longrightarrow \begin{bmatrix}1&2&3\\0&0&0\\0&2&2\\0&4&4\end{bmatrix} \longrightarrow \begin{bmatrix}1&2&3\\0&1&1\\0&0&0\\0&0&0\end{bmatrix} = U$

$$\rightarrow \begin{bmatrix}\boxed{\begin{matrix}1&0\\0&1\end{matrix}}&\boxed{\begin{matrix}1\\1\end{matrix}}\\ \begin{matrix}0&0\\0&0\end{matrix}&\begin{matrix}0\\0\end{matrix}\end{bmatrix} = R$$

"reduced row echelon form"

pivot cols (rank $r=2$) — free col.

Pivot vars: x_1, x_2

Free vars: x_3

Fact: A and A^T have the same rank!

Next step: Set free variable(s) to a convenient value

$$\begin{cases} x_1 \quad + x_3 = 0 \\ \quad x_2 + x_3 = 0 \end{cases} \qquad x_3 = 1 \implies x = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$$ "special sol'n"

Complete sol'n: $x = c\begin{bmatrix}-1\\-1\\1\end{bmatrix}$

Note that $x = c\begin{bmatrix}\boxed{\begin{matrix}-1\\-1\end{matrix}}\\ \boxed{1}\end{bmatrix}$ $-F$, I