8 Solving $Ax=b$

Example: $x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$

$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$

$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$

Note Row 1 + Row 2 = Row 3 $\Rightarrow b_1 + b_2 = b_3$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right]$$

"Augmented matrix" $[A \;\, b]$

pivot cols

$$\longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix} \Rightarrow 0 = b_3 - b_2 - b_1 \quad \left(\text{e.g., } b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}\right)$$

Solvability $Ax = b$ solvable when $b \in C(A)$.

If a combination of rows of A gives the zero row, then the same combin. of entries of b must give 0.

To find complete solution of $Ax=b$ (also called general sol'n).

① $x_{\text{particular}}$: - Set all free variables to zero

"any sol'n" - Solve $Ax=b$ for pivot variables.

Ex: Free vars $x_2 = x_4 = 0$:

$$x_1 + 2x_3 = 1, \quad 2x_3 = 3, \quad \text{say } b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \Rightarrow x_3 = \frac{3}{2},\ x_1 = -2.$$

Thus, $x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$ is a sol'n.

(2) $x_{\text{nullspace}}$: Find nullspace, x_n of A:

Ex: $x_n = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

(3) Add x_{part} & x_{null}:

$$\boxed{x = x_p + x_n}$$

1 particular sol'n — set of all sol'ns to $Ax = 0$.

Why this works:

$$\begin{aligned} A x_p &= b \\ + \quad A x_n &= 0 \\ \hline A(x_p + x_n) &= b \end{aligned}$$

Ex: $x = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

Overview of $Ax = b$ sol'ns

Let A be an $m \times n$ matrix of $\underbrace{\text{rank } r}_{\#\text{ of pivots}}$.

Note: $r \le m$ (at most 1 pivot per row)

$r \le n$ (at most 1 pivot per column)

(i) Full column rank: $\boxed{r=n}$. "every column has a pivot"

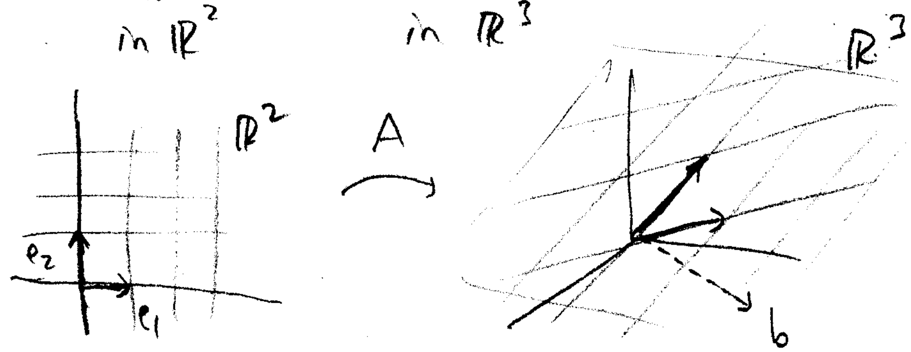
No free variables $\Rightarrow N(A) = \{\vec{0}\}$.

Sol'n to $Ax=b$ is $x = x_p$ (unique sol'n, <u>if</u> it exists)

<u>Ex</u>: $A = \begin{bmatrix}1 & 3\\ 2 & 1\\ 6 & 1\end{bmatrix}$ Map $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{bmatrix}1 & 3\\ 2 & 1\\ 6 & 1\end{bmatrix}\underset{\text{in } \mathbb{R}^2}{\begin{bmatrix}x_1\\ x_2\end{bmatrix}} = \underbrace{x_1\begin{bmatrix}1\\ 2\\ 6\end{bmatrix} + x_2\begin{bmatrix}3\\ 1\\ 1\end{bmatrix}}_{\text{in } \mathbb{R}^3}$$

RREF: $R = \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$



(ii) Full row rank: $\boxed{r=m}$ "every row has a pivot"

No zero rows in RREF $\Rightarrow$ no requirements on b

$\Rightarrow$ Can solve $Ax=b$ for <u>every</u> b.

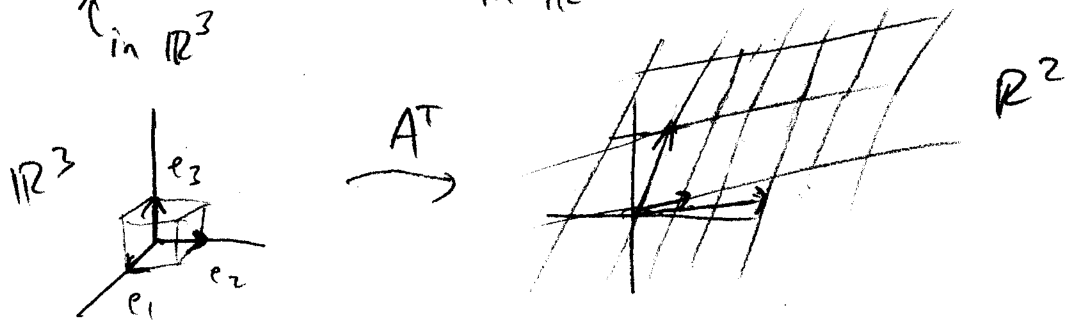
We're left with $n-r$ free variables $\Rightarrow$ Infinitely many sol'ns.

<u>Ex</u>: $A^T = \begin{bmatrix}1 & 2 & 6\\ 3 & 1 & 1\end{bmatrix}$. RREF $= \begin{bmatrix}1 & 0 & -\\ 0 & 1 & -\end{bmatrix}$

Map $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$\begin{bmatrix}1&2&6\\3&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \underbrace{x_1\begin{bmatrix}1\\3\end{bmatrix} + x_2\begin{bmatrix}2\\1\end{bmatrix} + x_3\begin{bmatrix}6\\1\end{bmatrix}}_{\text{in } \mathbb{R}^3}$$

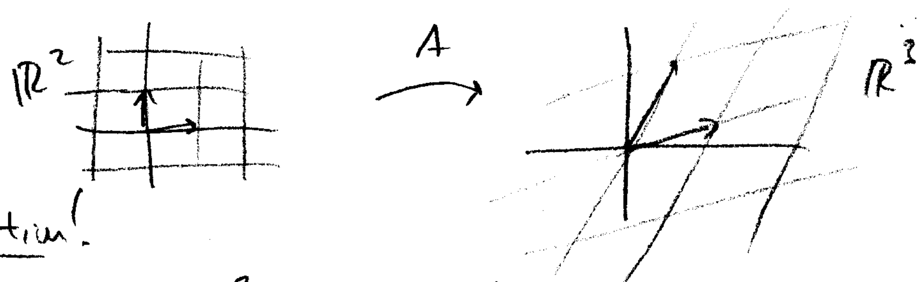
↑ in $\mathbb{R}^3$



(iii) Full rank: $\boxed{r=n=m}$ (A is <u>invertible</u>)

<u>Ex:</u> $A = \begin{bmatrix}1&2\\3&1\end{bmatrix}$ Map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (bijection)

<u>RREF</u>: $R = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

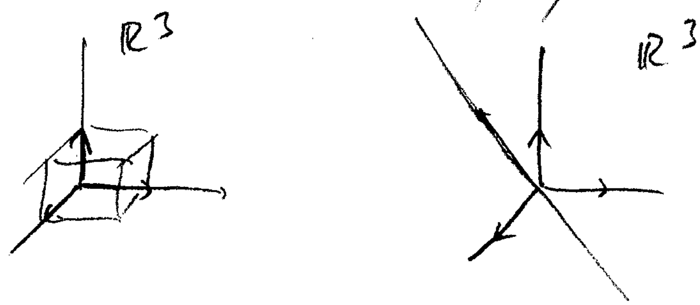


* $Ax=b$ has a <u>unique</u> <u>solution</u>!

(iv) $\boxed{r<m, \ r<n}$

$$R = \begin{bmatrix}I&F\\0&0\end{bmatrix}$$

<u>Ex:</u>



* $Ax=b$ has 0 or ∞ sol'ns.

<u>Summary:</u>

up to perm. of cols.

<u>$r=m=n$.</u>	<u>$r=n<m$</u>	<u>$r=m<n$</u>	<u>$r<m, \ r<m$</u>
$R=I$	$R=\begin{bmatrix}I\\0\end{bmatrix}$	$R=\begin{bmatrix}I & F\end{bmatrix}$.	$R=\begin{bmatrix}I&F\\0&0\end{bmatrix}$
Unique sol'n to $Ax=b$	0 or 1 sol'n to $Ax=b$.	∞ sol'ns to $Ax=b$.	0 or ∞ sol'ns to $Ax=b$.