II. Other vector spaces

Let $M_{3\times3}$ = set of all 3×3 matrices.

This is a vector space: $A+B$, cA are 3×3 matrices if A, B are.

(ignore AB for now)

Basis: $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \dots, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$.

Subspaces of $M_{3\times3}$:

- U = upper triangular matrices $\begin{bmatrix}a&b&c\\0&d&e\\0&0&f\end{bmatrix}$ $\dim U = 6$
- S : Symmetric matrices : $\begin{bmatrix}a&b&c\\b&d&e\\c&e&f\end{bmatrix}$ $\dim S = 6$
- $S\cap U$ = diagonal matrices. $\begin{bmatrix}a&0&0\\0&d&0\\0&0&f\end{bmatrix}$ $\dim S\cap U = 3$

Note: $S\cup U$ is not a subspace. (why?)

But the sum $S+U = \{s+u : s\in S, u\in U\}$ is a subspace.

$= $ all 3×3 matrices. (why?)

$*$ $\dim(S+U) = \dim S + \dim U - \dim(S\cap U)$.

2

<u>Def</u>: A <u>linear</u> map between vector spaces satisfies

$$\boxed{f(cv+dw) = c\,f(v) + d\,f(w)} \quad \text{for all } c, d, v, w$$

An $m \times n$ matrix is a linear map $A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Indeed: $A(cv+dw) = c\,Av + d\,Aw$.

<u>EX</u>: $\mathcal{C}^\infty$ is the vector space of smooth functions. (∞ diff'ble).

Consider the equation $\boxed{\dfrac{d^2y}{dx^2} + y = 0}$ $(*)$.

Which can be written as $\underbrace{\left(\dfrac{d^2}{dx^2} + 1\right)}_{\text{"linear map"}} y = 0$.

The set of solutions to $(*)$ is the <u>nullspace</u> of the "linear differential operator" $D := \dfrac{d^2}{dx^2} + 1$. ($2^{nd}$ order).

<u>Fact</u>: The nullspace, $y_n(x)$, is 2-dimensional

<u>Note</u>: $y_1(x) = \cos x$ and $y_2(x)$ are solutions. (i.e., in $N(D)$).

So $\{\cos x, \sin x\}$ is a <u>basis</u> for the solution space.

i.e, $y_n(x) = C_1 \cos x + C_2 \sin x$ is the "general solution".

Ex 2: Now consider $y'' + y = 10$

i.e, $$\underbrace{\left(\frac{d^2}{dx^2} + 1\right)}_{A} \underset{\vec{x}}{y} = \underset{\vec{b}}{10}$$

Soln: $y = y_p + y_n$

$\uparrow$ $y_p(x) = 10$, by inspection!

General soln: $y(x) = C_1 \cos x + C_2 \sin x + 10$.

Rank-one matrices

Ex: $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$ Basis for row space: $\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

Basis for col. space $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Recall: $\dim C(A) = \text{rank } A = \dim C(A^T)$.

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$$

$\uparrow$ pivot col

\# Every rank-1 matrix has the form $A = uv^T$.

Rank-1 matrices are the "building blocks"

Rank-k matrices are sums of at least k rank-1 matrices.

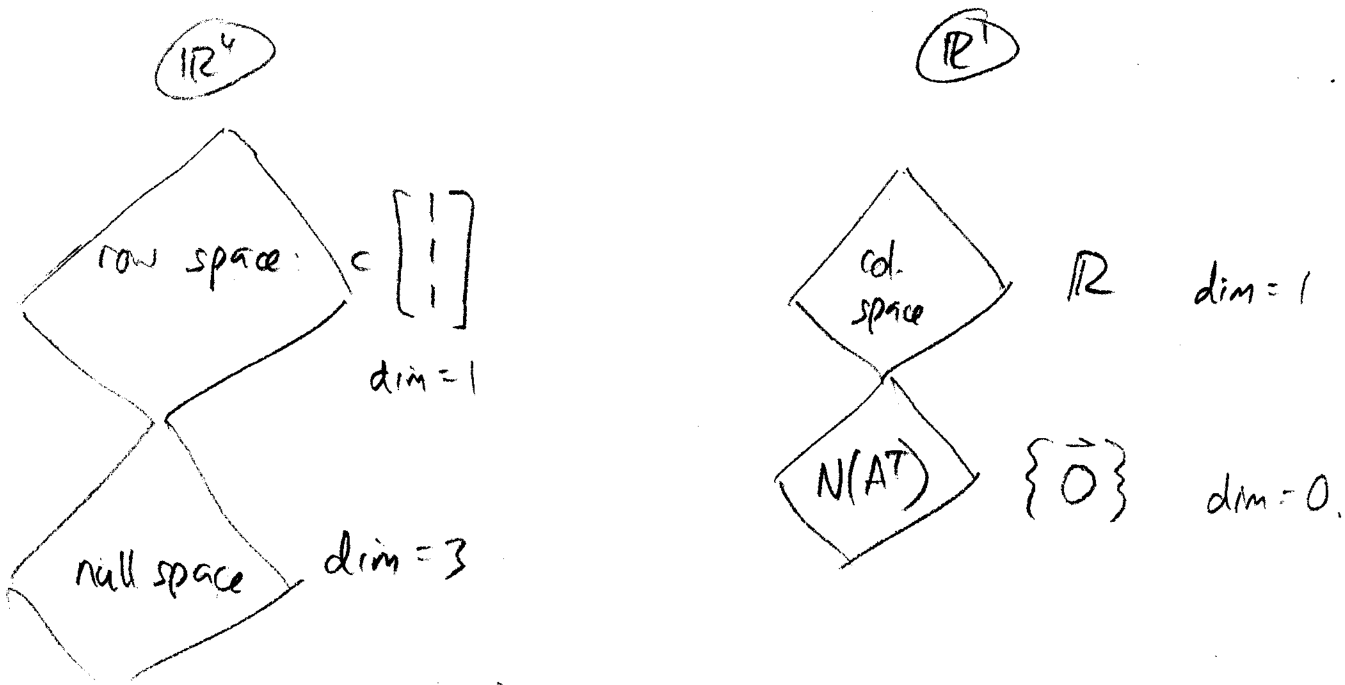
<u>Ex:</u> In $\mathbb{R}^4$, $S = \{\vec{v} : v_1 + v_2 + v_3 + v_4 = 0\}$

$= \text{nullspace of } A = [1 \ 1 \ 1 \ 1]$

rank $A = 1 = r$

$\dim N(A) = n - r = n - 1$

Fundamental subspaces of A:



Basis: $\left\{ \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix} \right\}$