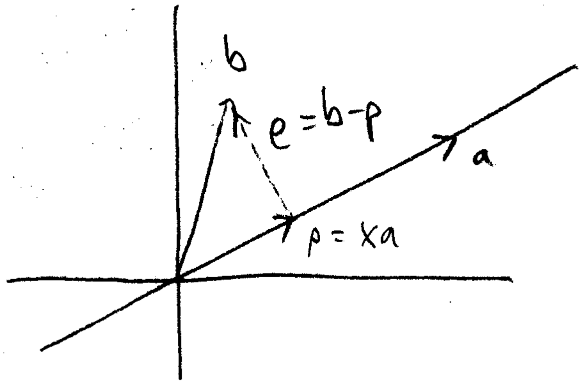
$\boxed{2}$ Projections



$$b = \underset{\text{projection onto } a}{p} + \underset{\text{error}}{e}$$

Goal: Write $\vec{a}^T(\vec{b} - x\vec{a})$ (i.e., $a \perp e$).

$$\Rightarrow \quad x a^T a = a^T b \quad \Rightarrow \quad \boxed{x = \frac{a^T b}{a^T a}} \qquad \left(= \frac{a \cdot b}{a \cdot a} \right)$$

[**Note:** If $\|n\| = 1$, then $\text{proj}_n v = v \cdot n$]

Question: What matrix P sends $\vec{b} \longmapsto \vec{p}$?

Ans: $$Pb = p = ax = a\left(\frac{a^T b}{a^T a}\right) = \frac{a a^T}{a^T a} b$$

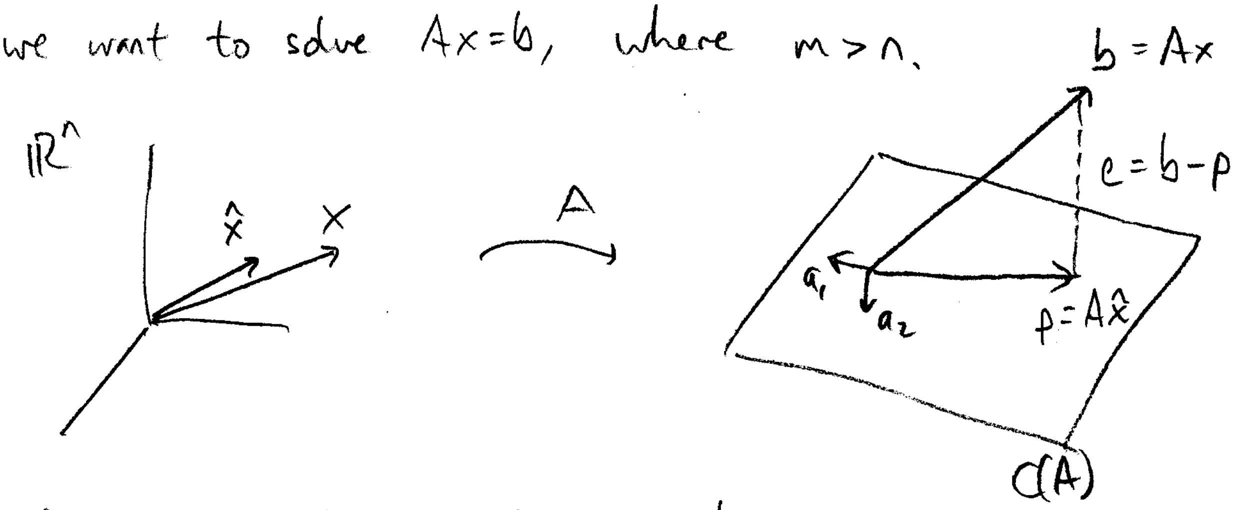
That is, $$P = \frac{\overset{\leftarrow \text{ rank 1 matrix}}{a a^T}}{\underset{\leftarrow \text{ number}}{a^T a}}$$

Remarks:

- $C(P)$ = line through a
- rank $P = 1$
- $P^T = P$
- $P^2 = P$ (defining property of projection matrices!)

Motivation:

Suppose we want to solve $Ax=b$, where $m>n$.



In general, there's no sol'n (because b won't lie in $C(A)$.)

But $p := \text{proj}_{C(A)} b$ is the closest vector to b that yields a sol'n. New goal: Solve $\boxed{A\hat{x} = p}$

How: Amazingly, this $\hat{x}$ is also the sol'n to $\boxed{A^TA\hat{x} = A^Tb}$!

Why: Suppose $C(A)$ is 2D, with basis a_1, a_2.

Write $A\hat{x} = p = \hat{x}_1 a_1 + \hat{x}_2 a_2$.

Remark: Whatever $\hat{x}$ is, $\underbrace{b - A\hat{x}}_{=e} \perp C(A)$ (i.e., $\perp$ to a_1 & a_2)

$$\Rightarrow \quad a_1^T(b - A\hat{x}) = 0 \quad , \quad a_2^T(b - A\hat{x}) = 0.$$

In matrix form: $\underbrace{\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}}_{A^T}(b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow A^T b - A^T A \hat{x} = 0 \quad \checkmark$$

Remark: If A^TA is invertible, then $\boxed{\hat{x} = (A^TA)^{-1}A^Tb}$

Claim: This happens whenever A has independent columns!

i.e, full row rank.

Proof: Suppose $A^TAx = 0$ (Goal: show $x = 0$).

$$\Rightarrow x^TA^TAx = 0$$

$$\Rightarrow (Ax)^TAx = 0$$

$$\Rightarrow \|Ax\| = 0 \Rightarrow x = 0 \text{ (since } A \text{ has indep cols.)} \checkmark$$

Big idea: Take a subspace S.

Suppose columns of A are a basis for S.

Then $A\hat{x}$ projects $\hat{x}$ onto S.

i.e, $A\hat{x} = A(A^TA)^{-1}A^T\hat{x}$

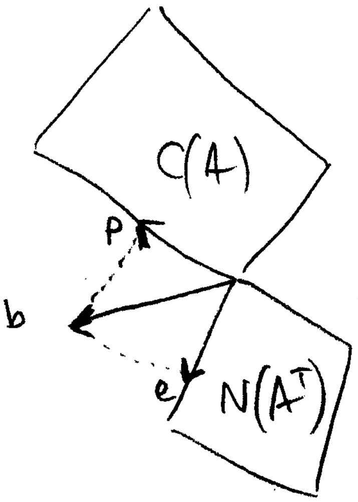
so $\boxed{P = A(A^TA)^{-1}A^T}$ is the projection matrix!

Compare to 1D: $P = \frac{aa^T}{a^Ta}$.

Check: $P^T = P$: $(A(A^TA)^{-1}A^T)^T = \overset{=A}{A^{TT}}\overset{\text{Symm.}}{(A^TA^{-1})^T}A^T$ ✓

$P^2 = P$: $A(A^TA)^{-1}\boxed{A^TA}(A^TA)^{-1}A^T$ ✓

Extreme cases:



① If $b \in C(A)$, $Pb = b$

Why: Say $b = Ax$.

$Pb = [A(A^TA)^{-1}A^T]Ax = Ax = b$ ✓

② If $b \perp C(A)$, $Pb = 0$

Why: $b \in N(A^T) \Rightarrow A^Tb = 0 \Rightarrow A(A^TA)^{-1}A^Tb = 0$. ✓

Remark: If $b = \underset{Pb}{p} + \underset{(I-P)b}{e}$, then $I-P$ projects onto ⊥ space.