4 Orthogonal bases

Def: A set $\{q_1, \dots, q_n\}$ of vectors is orthonormal if

$$q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases} \qquad (\text{recall } \|q_i\|^2 = q_i^T q_i.)$$

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}, \quad Q^T Q = \begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Def: An orthogonal matrix is a square matrix Q such that $\boxed{Q^T Q = I}$.

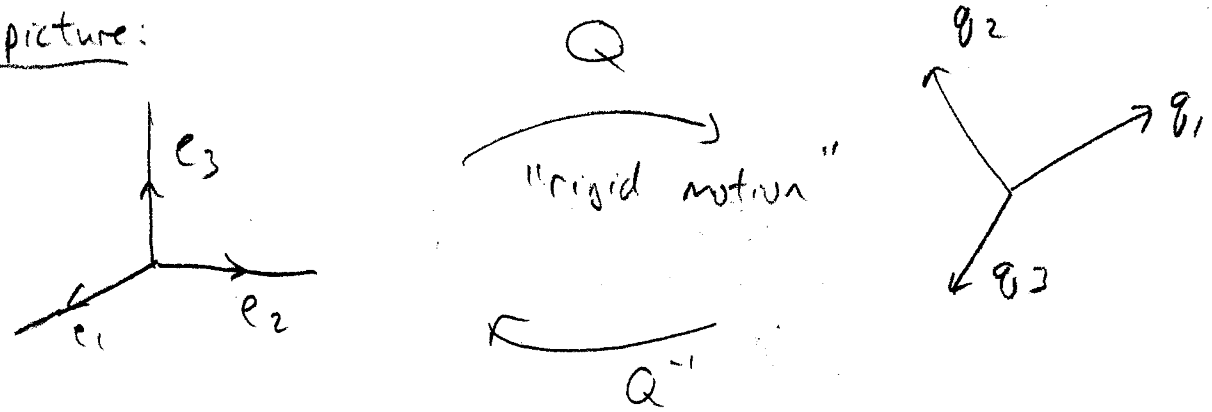
Note: In this case, $Q^T = Q^{-1}$.

Ex: • Permutation matrices $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

• $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

• $Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ "Adamar Matrix" $\quad Q = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$

Grid picture:



Geometrically, Q is a rigid motion; composed of rotations and/or reflections.

Preserves lengths: $\|Qx\|^2 = (Qx)^T(Qx) = x^TQ^TQx = x^Tx = \|x\|$.

Preserves angles: $(Qx)^T(Qy) = x^TQ^TQy = x^Ty$ ✓

Other nice properties:

- If Q has orthonormal columns (need not be square), then $P = Q(Q^TQ)^{-1}Q^T = QQ^T$ is the projection matrix onto $C(Q)$. (And $QQ^T = I$ if Q is square).
- If Q is square, $b = QQ^Tb \Rightarrow \boxed{b = q_1(q_1^Tb) + \dots + q_n(q_n^Tb)}$.
- least squares: $Q^TQ\hat{x} = Q^Tb \Rightarrow \hat{x} = Q^Tb = \begin{bmatrix} q_1^Tb \\ \vdots \\ q_n^Tb \end{bmatrix}$.

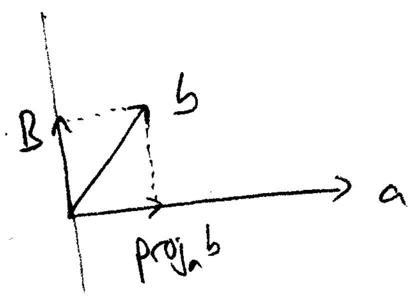
Gram-Schmidt process.

Input: Independent set of vectors

Output: Orthonormal set.

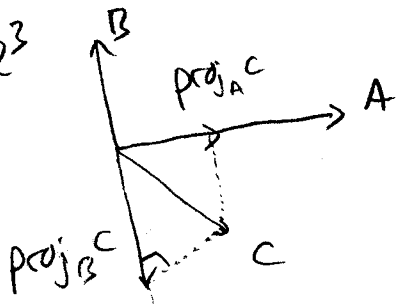
Idea: $a, b, c \xrightarrow{\text{(G-S)}} A, B, C \xrightarrow{\text{(normalize)}} q_1, q_2, q_3$

independent $\quad$ orthogonal $\quad$ orthonormal

Ex. 

$$A = a$$

$$B = b - \mathrm{proj}_A\, b = b - \frac{A^T b}{A^T A} A$$

$\mathbb{R}^3$ 

$$C = c - \mathrm{proj}_A\, c - \mathrm{proj}_B\, c$$

$$= c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B.$$

Check: $A^T B = A^T\left(b - \frac{A^T b}{A^T A} A\right) = 0$ ✓

Last step: $q_1 = \frac{A}{\|A\|}, \quad q_2 = \frac{B}{\|B\|}, \quad q_3 = \frac{C}{\|C\|}.$

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Ex: $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$

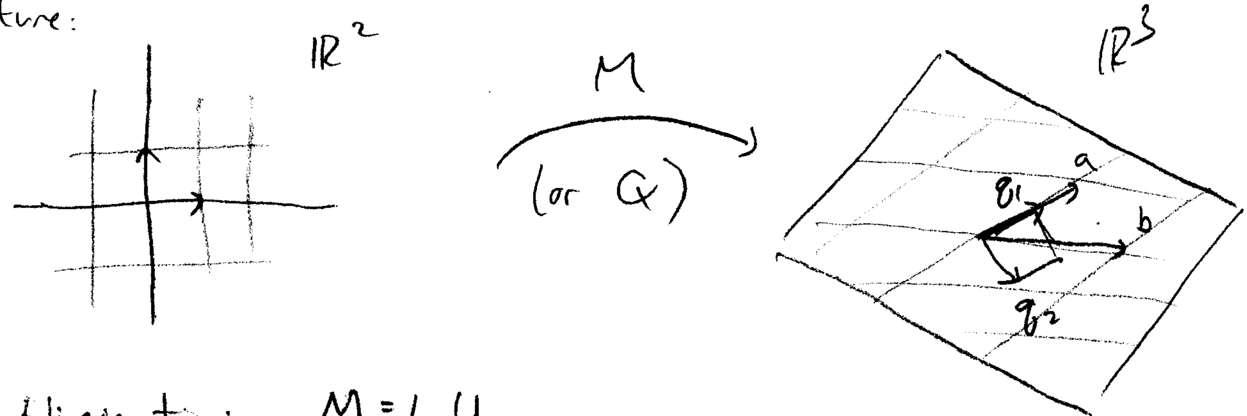
$$A = a. \qquad B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{(1,1,1)\cdot(2,0,4)}{(1,1,1)\cdot(1,1,1)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}.$$

$$Q = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}.$$

$\hookrightarrow$ Same col. space as $M := \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$!

Grid picture:



Recall: Elimination: $M = LU$

Gram-Schmidt: $M = QR$ $\leftarrow$ upper-triangular!

$$\begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ q_2^T a & q_2^T b & q_2^T c \\ q_3^T a & q_3^T b & q_3^T c \end{bmatrix}$$

$= 0$ (why?)