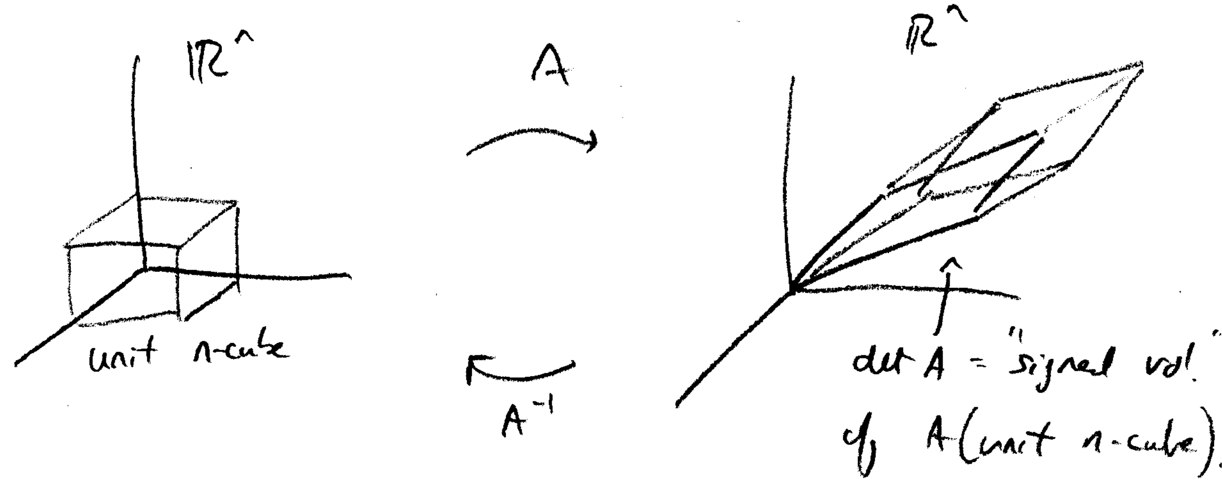
15 Determinants

A determinant is a number associated with a square matrix.

It represents the scaling factor in the "grid picture."

Preview:



Notation: $\det A$ or $|A|$.

* Keep referring back to this picture!

Properties

① $\det I = 1$ $\quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

② Exchange rows: reverse sign of det. $\quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

(so $\det P = \pm 1$ for perm. matrices.

③ det is a linear function of a fixed row.

(3a) $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ (pull out constants)

(3b) $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ (break apart sums)

Caution: $\det kA \neq k \det A$

$\det(A+B) \neq \det A + \det B$.

* Properties (4) – (10) all follow from (1) – (3).

(4) 2 equal rows $\Rightarrow \det A = 0$.

Why: Exchanging 2 rows flips sign of det, preserves the matrix.

So $\det A = -\det A \Rightarrow \det A = 0$.

(5) $\underbrace{\text{Subtract } l \times (\text{row } i) \text{ from row } k}_{\text{"elementary row operation"}}$: det doesn't change!

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{=0} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Works similar for $n \times n$.

(6) Row of zeros $\Rightarrow$ $\det A = 0$.

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 0\cdot a & 0\cdot b \\ c & d \end{vmatrix} = 0 \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0.$$

(7) $$\det \begin{bmatrix} d_1 & & & * \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} = d_1 d_2 \cdots d_n \quad \text{(product of pivots)}.$$

Why: $\xrightarrow[\text{row opers.}]{\text{Elementary}}$ $$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} = D \text{ if no } d_i = 0.$$

$\det D = d_1 d_2 \cdots d_n$ (prop. 3a).

If some $d_i = 0$, then by elimination, we can make one row all zeros $\Rightarrow$ $\det D = 0$.

(8) $\det A = 0$ exactly when A is singular.

(immediate from (7)).

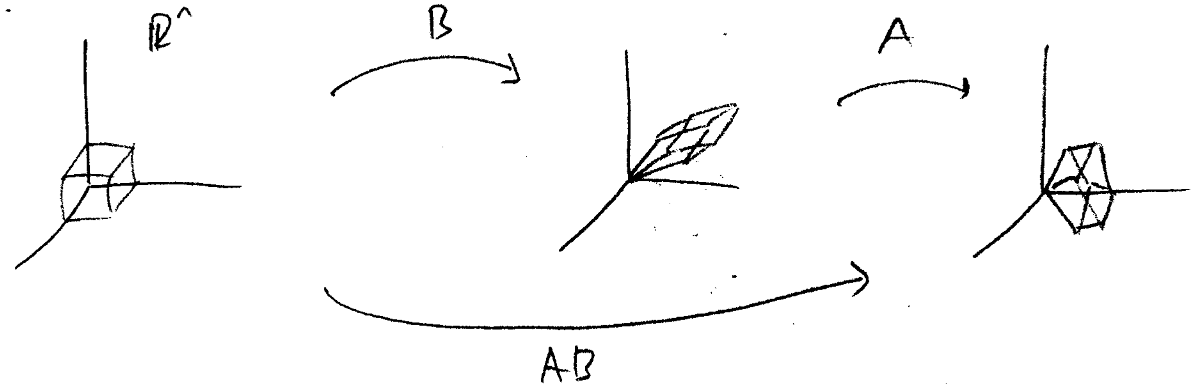
Ex: $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{vmatrix} = a\left(d - \frac{c}{a} b\right) = ad - bc.$$

(9) $\det(AB) = (\det A)(\det B)$.

(No proof; it's hard)

(4)

Intuition:



Scaling factor of "B then A" = (scaling factor of A)(scal. fact. of B).

Consequences:

- $1 = \det I = \det AA^{-1} = \det A \det A^{-1} \Rightarrow \det A^{-1} = \dfrac{1}{\det A}$
- $\det(A^2) = (\det A)^2$
- $\det(2A) = 2^n \det A$ (actually from (3a)).

(10) $\det A^T = \det A$.

* So all "row properties" are also "column properties."

Proof of (10):

$$|A| = |LU| = |L|\cdot|U| = |U| = |D|$$

$$|A^T| = |U^T L^T| = |U^T|\,|L^T| = |U^T| = |D|.$$

$$A = \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{bmatrix}}_{U}$$