6 Formulas for det A

2×2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

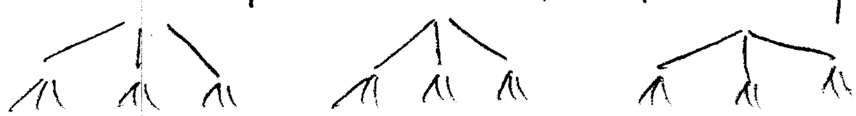
$$= \left(\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} \right) + \left(\begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} \right)$$

4 pieces, but 2 non-zero

$$= 0 + ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 0 = \boxed{ad - bc}$$

3×3

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ & - & \\ & - & \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ & - & \\ & - & \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{12} \\ & - & \\ & - & \end{vmatrix}$$



Non-zero terms: (1 for each permutation matrix)

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Formula: $\det A = \sum_{n!\ \text{terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$

where $(\alpha, \beta, \gamma, \ldots, \omega)$ is a permutation of $(1, 2, \ldots, n)$

Ex:

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= -2 + 1 = -1.$$

Cofactors:

Recall:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

Cofactor of $a_{ij} := C_{ij} = \pm \det \begin{pmatrix} n-1 \text{ matrix with row } i, \\ \text{col. } j \text{ erased} \end{pmatrix}$

$$\begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}$$

Cofactor formula:

$$\boxed{\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}}$$

* works for any <u>row</u> $i = 1, 2, \dots, n$.

We can also expand cofactors across a <u>column</u> $j = 1, 2, \dots, n$:

$$\boxed{\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}}$$

Check 2×2 case: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a(d) + b(-c) = ad - bc.$ ✓

Ex: $\begin{vmatrix} 1 & 0 & 3 \\ -4 & 2 & 1 \\ -2 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - 0\begin{vmatrix} -4 & 1 \\ -2 & 2 \end{vmatrix} + 3\begin{vmatrix} -4 & 2 \\ -2 & 0 \end{vmatrix}$ (row 1)

$= 1 \cdot 4 - 0(-6) + 3 \cdot 4 = 16$

OR: Column 2: $= -0 + 2\begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} - 0 = 2 \cdot 8 = 16.$ ✓

Example: Consider the following triagonal matrices:

$A_1 = [1]$ $|A_1| = 1$

$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $|A_2| = 0$

$A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $|A_3| = 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$

$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $|A_4| = 1 \cdot |A_3| - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ ← cofactor along column 1

$= |A_3| - |A_2| =$

* In general: $|A_n| = |A_{n-1}| - |A_{n-2}|$

$|A_1| = 1,\ |A_2| = 0,\ |A_3| = -1,\ |A_4| = -1,\ |A_5| = 0,\ |A_6| = 1$

$|A_7| = 1,\ |A_8| = 0,\ |A_9| = -1, \ldots$ [periodic with period 6.]