7 Applications of determinants

1. Formula for A^{-1}
2. Cramer's rule for $x = A^{-1}b$
3. $|\det A|$ = vol. of box

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① Motivation (2x2 case): $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\det A}} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\text{co-factors}} = \frac{1}{\det A} \begin{bmatrix} C_a & C_c \\ C_b & C_d \end{bmatrix}.$

Formula: $\boxed{A^{-1} = \frac{1}{\det A} C^T}$, where $C = (\underbrace{C_{ij}}_{\text{cofactor of } a_{ij}})$

Check (by showing $AC^T = (\det A)\, I$.)

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & & 0 \\ & \det A & & \\ 0 & & \ddots & \\ & & & \det A \end{bmatrix}$$

Diagonal elts = det A by cofactor formula:

$\det A = a_{11}C_{11} + \cdots + a_{n1}C_{n1}$

Why are off-diagonal elts zero?

Compare: $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ vs. $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$

$$= \begin{vmatrix} a_{11} & a_{22} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{vs.} \quad = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

(2) Solving $Ax = b$ (A inv'l): $\quad x = A^{-1}b = \frac{1}{\det A} C^T b$

Cramer's rule:

$$x_1 = \frac{\det B_1}{\det A} \quad \text{where} \quad B_1 = \begin{bmatrix} b & \begin{matrix} n-1 \text{ cols} \\ \text{of } A \end{matrix} \end{bmatrix}$$

$$x_2 = \frac{\det B_2}{\det A} \qquad B_2 = \begin{bmatrix} | & b & - \end{bmatrix}$$

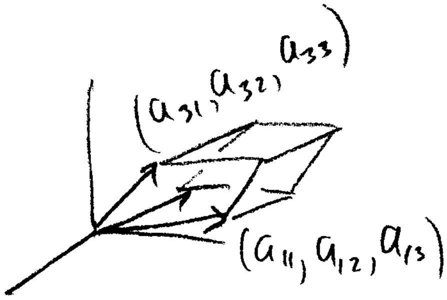
$$\vdots \qquad\qquad \vdots$$

$$x_n = \frac{\det B_n}{\det A} \qquad B_n = \begin{bmatrix} \begin{matrix} n-1 \text{ cols} \\ \text{of } A \end{matrix} & b \end{bmatrix}$$

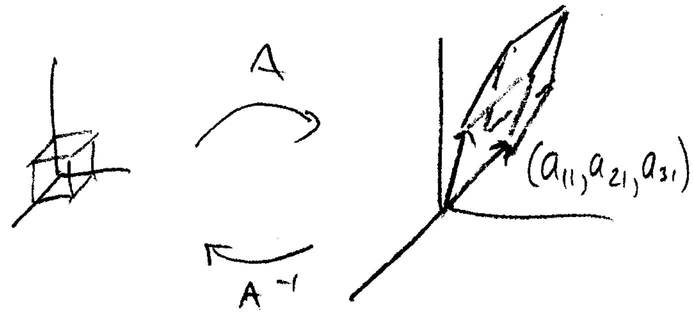
Note: $\det B_1 = b_1C_{11} + b_2C_{21} + \dots + b_nC_{n1}$ $\quad$ ($B_i = A$ with col. i replaced by b).

(3) $|\det A|$ = vol. of box ("parallelpiped") formed by the row vectors (or col. vectors.)

"row box"



"column box"

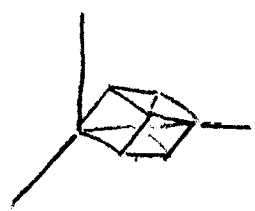


the "grid picture"

* These have the same volume because $|A| = |A^T|$.

<u>Ex:</u>

- $\det I = 1$ ✓
- $\det Q = \pm 1$ (Q orthogonal)



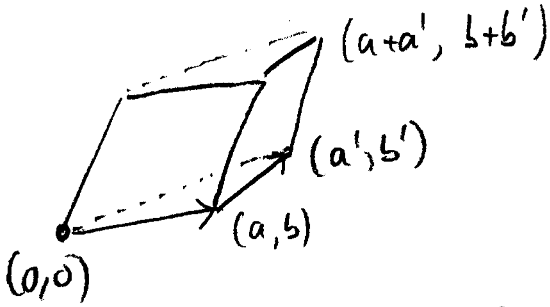
- If A has orthogonal cols $v_1, \ldots, v_n$, then $|\det A| = \|v_1\| \cdot \|v_2\| \cdots \|v_n\|$ (Why?)

<u>Check</u> properties:

1. $\det I = 1$ ✓
2. Swap two rows preserves $|\det A|$ ✓

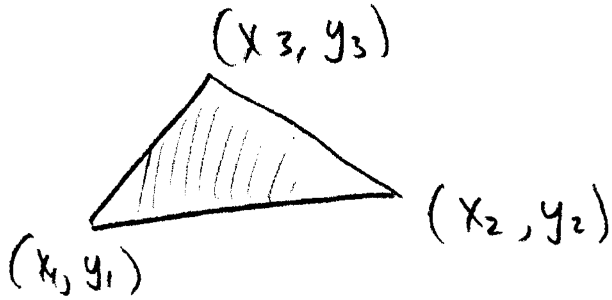
3a. Scale a row scales $\det A$ by k ✓

(3b) e.g.,
$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$



✓

<u>Area of a triangle</u> :



$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right)$$