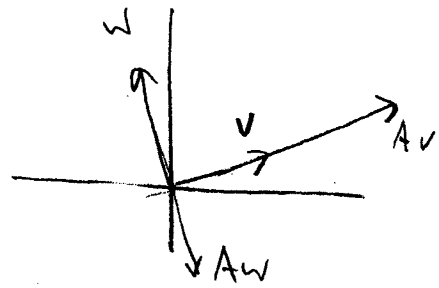
8 Eigenvalues and Eigenvectors

Def: If $\boxed{Av = \lambda v}$ (where λ is a scalar), then λ is an eigenvalue of A and v is an eigenvector.

Motivation: "grid picture"

$\mathbb{R}^n$ $\xrightarrow{A}$ $\mathbb{R}^n$ Ax (not an e-vector)

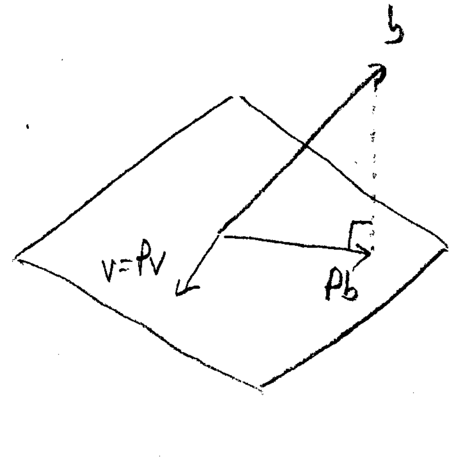
Want: Av "parallel" to v:



Special case: If A is singular, $\lambda = 0$ is an e-value.

Any vector v in $N(A)$ is an e-vector.

Ex: Suppose P is a projection matrix.



Eigenvectors:

- Any v in plane
  $Pv = v$, so $\boxed{\lambda = 1}$
- Any $v \perp$ plane
  $Pv = 0$, so $\boxed{\lambda = 0}$

<u>Ex:</u> $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Av_1 = v_1 \Rightarrow \boxed{\lambda_1 = 1}$

$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $Av_2 = -v_2 \Rightarrow \boxed{\lambda_2 = -1}$

<u>Solving $Av = \lambda v$:</u>

Rewrite $\underbrace{(A - \lambda I)}_{\text{must be singular!}} v = 0 \Rightarrow \boxed{\det(A - \lambda I) = 0}$

Find λ first:

Then plug in λ and find nullspace of $A - \lambda I$.

<u>Ex:</u> $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$$

$$= \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2) = 0$$

$$\Rightarrow \boxed{\lambda_1 = 4, \quad \lambda_2 = 2}$$

Find e-vectors:

$\boxed{\lambda_1 = 4}$: $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (in $N(A - 4I)$.)

$\boxed{\lambda_2 = 2}$. $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (in $N(A - 2I)$.)

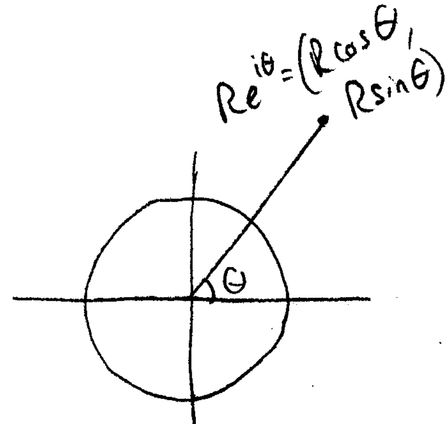
Remark: Compare $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Eigenvectors: Same.

Eigenvalues: differ by 3.

Why: If $Av = \lambda v$, then $(A+3I)v = Av + 3v = \lambda v + 3v = (\lambda+3)v$.

Ex (Complex valued e-values):



θ in radians.

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{(90° rotation matrix)}$$

$$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\Rightarrow \boxed{\begin{matrix} \lambda_1 = i \\ \lambda_2 = -i \end{matrix}} \begin{matrix} = e^{i\pi/2} \\ = e^{-i\pi/2} \end{matrix} \quad \text{(complex conjugate pairs)}$$

Recall: $(R_1 e^{i\theta_1})(R_2 e^{i\theta_2}) = R_1 R_2 e^{i(\theta_1+\theta_2)}$ ← angles add.

↑ lengths multiply

So multiplying by $Re^{i\theta}$ scales by R, rotates by θ.

Eigenvectors of Q:

$\boxed{\lambda_1 = i}$ $\quad Q - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$ $\quad \begin{matrix} -ix_1 - x_2 = 0 \\ x_1 - ix_2 = 0 \end{matrix}$ scalar multiples

$\Rightarrow x_1 = i x_2 \Rightarrow v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\boxed{\lambda_1 = -i}$ $\quad$ Similarly, $v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

Ex: Repeated eigenvalues, shortage of eigenvectors.

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \quad \det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda_1 = 3, \quad \lambda_2 = 3.$$

Eigenvectors: $(A-3I) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$N(A-3I) = c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Only 1 e-vector!

Ex: Repeated eigenvalues, no eigenvector shortage.

$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. $\lambda_1 = \lambda_2 = 3$, $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(actually, every vector is an e-vector!)

Remark: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\chi_A(\lambda) := \det(A-\lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - \underbrace{(a+d)}_{\text{"trace of A"; tr } A}\lambda + \underbrace{(ad-bc)}_{\det A}$$

Facts: $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$ (product of eigenvalues)

$\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ (sum of eigenvalues)

also equal to sum of diagonal entries.

Why: Consider $\chi_A(\lambda) = \det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$.