10 Application: Markov chains

Markov matrices:

① All entries ≥ 1

② All cols. add to 1

Ex: $A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$

Key properties

(i) $\lambda_1 = 1$ is an e-value

(ii) All other e-values are $|\lambda_i| \leq 1$ ("usually" $|\lambda_i| < 1$)

(iii) E-vector v_1 has non-negative entries.

For any u_0,

$$u_k = A^k u_0 = \underbrace{c_1 \lambda_1^k v_1}_{\to\, c_1 v_1 \text{ (steady-state)}} + \underbrace{c_2 \lambda_2^k v_2 + \cdots + c_n \lambda_n^k v_n}_{\to\, 0 \text{ if } |\lambda_i| < 1}$$

Why is $\lambda = 1$ an e-value?

$A - 1I = \begin{bmatrix} -.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix}$

Cols. add to zero

$\Rightarrow$ row 1 + row 2 + row 3 = 0

$\Rightarrow$ $A - 1I$ is singular.

Remark: $(1,1,1)$ is in $N((A-I)^T)$

e-vector v_1 is in $N(A-I)$.

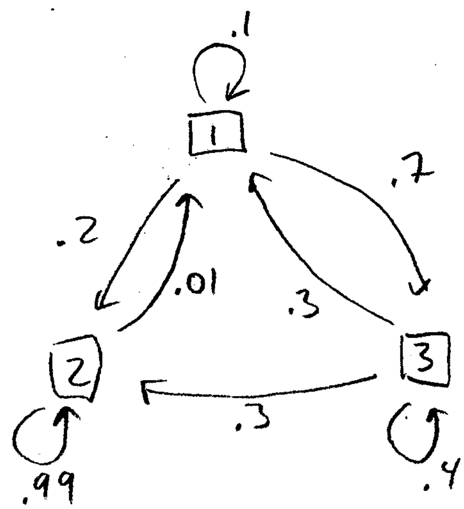
Note: A and A^T have the same e-values.

Why: $\det(A-\lambda I)=0$ & $(A-\lambda I)^T = A^T - \lambda I$

$\Rightarrow \det(A^T - \lambda I) = 0$.

A Markov matrix (or Markov chain) A is a matrix of "transition probabilities."

Ex: $A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$



Questions: Given an initial state $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

what is the long-term behavior, i.e., $\lim_{k\to\infty} A^k u_0$?

Ex: $\begin{bmatrix} u_w \\ u_e \end{bmatrix}_{t=k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_w \\ u_e \end{bmatrix}_{t=k}$

u_w = # people on west coast

u_e = # people on east coast.

Check: $\lambda_1 = 1$ (always) $\Rightarrow \lambda_2 = .7$.

$$A - I = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{v_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad A - .7I = \begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{v_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Initial state: Let $u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = \underbrace{c_1}_{1000/3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \underbrace{c_2}_{2000/3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$ "# of people on each coast in year k"

Steady-state: $$\lim_{k \to \infty} u_k = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2000/3 \\ 1000/3 \end{bmatrix} = \begin{bmatrix} 666.7 \\ 333.3 \end{bmatrix}$$