I Symmetric matrices

Properties: (if $A^T = A$):

1. Eigenvalues are real.
2. A has n orthogonal eigenvectors.

Usual case: $A = S\Lambda S^{-1}$

Symmetric: $A = Q\Lambda Q^{-1} = \boxed{Q\Lambda Q^T}$ "spectral theorem", or "principal axis theorem"

Why:

Real e-values: (Assume A is real)

$$Av = \lambda v \Rightarrow A\bar{x} = \bar{\lambda}\bar{x} \qquad \text{Take complex conjugate}$$
$$\Rightarrow \bar{x}^T A^T = \bar{x}^T \bar{\lambda} \qquad \text{Transpose both sides}$$
$$\Rightarrow \bar{x}^T A = \bar{x}^T \bar{\lambda} \qquad A^T = A$$
$$\Rightarrow \bar{x}^T A x = \bar{x}^T \bar{\lambda} x \qquad \text{right multiply by } x$$
$$\Rightarrow \bar{x}^T \lambda x = \bar{x}^T \bar{\lambda} x \qquad \text{since } Ax = \lambda x$$
$$\Rightarrow \lambda(\bar{x}^T x) = \bar{\lambda}(\bar{x}^T x)$$
$$\Rightarrow \lambda = \bar{\lambda} \quad (\text{assuming } \bar{x}^T x \neq 0 \text{ for } x \neq 0.)$$

Recall: $(a+bi)\overline{(a+bi)} = (a+bi)(a-bi) = a^2 + b^2 = |a+bi|^2$.

Why $\bar{x}^T x \neq 0$:

$$\bar{x}^T x = [\bar{x}_1 \ \bar{x}_2 \ \cdots \ \bar{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 > 0$$

<u>Orthogonal e-vectors</u>

<u>Claim</u>: $A^T = A$, & $Av = \lambda v$, $Aw = \mu w$ with $\lambda \neq \mu$, then $\boxed{v \perp w}$

<u>Proof</u>: $\lambda v^T w = (Av)^T w = v^T A^T w = v^T A w = \mu v^T w$

$\Rightarrow \lambda v^T w = \mu v^T w$ and $\lambda \neq \mu \Rightarrow v^T w = 0$. ✓

<u>Remark</u>: If A has <u>complex</u> entries, then the above results hold if $A^T = A$ is replaced by $\boxed{\bar{A}^T = A}$: "Hermitian."

<u>Summary</u>: If $A^T = A$, then $A = Q \Lambda Q^T$:

$$A = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} — & q_1^T & — \\ — & q_2^T & — \\ & \vdots & \\ — & q_n^T & — \end{bmatrix}$$

$$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T \leftarrow \text{projection matrix!}$$

* Every symmetric matrix is a <u>linear combination</u> of <u>projection</u> matrices onto orthogonal axes.

Compututing e-values of large matrices is hard!

<u>Useful fact</u>: For symmetric matrices:

positive e-values = # positive pivots

negative e-values = # negative pivots.

Recall that e-values of $A + cI$ are c plus e-values of A.

Pivots are easy to compute (for large matrices).

These facts are useful for designing <u>numerical algorithms</u> for computing e-values.

<u>Example</u>: (# pos. pivots of $A + 10I$) - (# pos. pivots of A)

= # of e-values of A s.t. $0 < \lambda < 10$.