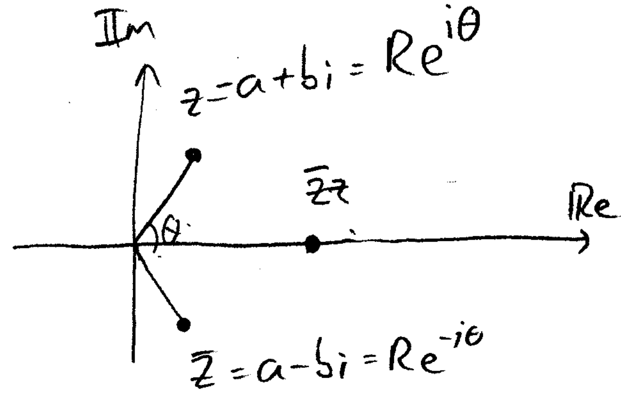
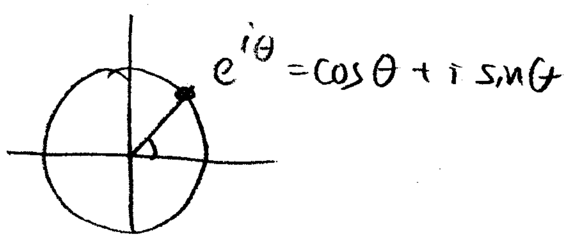
3 Complex matrices

Review of complex numbers:



let $z=a+bi$

Recall: $e^{i\theta}$ is on the unit circle:



$$|z|^2=z\bar{z}=a^2+b^2$$

$$(R_1e^{i\theta_1})(R_2e^{i\theta_2})=R_1R_2e^{i(\theta_1+\theta_2)}$$ "lengths multiply; angles add"

Now, consider $z=\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ in $\mathbb{C}^n$.

$$|z|^2=\bar{z}^Tz=\begin{bmatrix}\bar{z}_1 & \bar{z}_2 & \cdots & \bar{z}_n\end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}=|z_1|^2+|z_2|^2+\cdots+|z_n|^2.$$

Inner product: $\langle x,y\rangle=y^Tx$ for real vectors

$\langle z,w\rangle=\bar{w}^Tz$ for complex vectors.

Notation: $w^Hz=\bar{w}^Tz$. Hermitian product

(2)

"Symmetry" for complex matrices: $\bar{A}^T = A$ ("Hermitian")

$$A^H = A$$

Ex: $\begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$ is Hermitian.

Orthogonality: $q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ for real vectors

$q_i^H q_j = \bar{q}_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ for complex vectors.

For matrices: $Q^T Q = I$ for real matrices (orthogonal)

$\bar{Q}^T Q = I$ for complex matrices

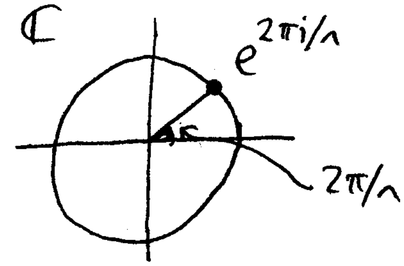
$Q^H Q = I$, called unitary.

Application Fast Fourier Transform (FFT)

Fourier matrix:

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix}$$

$$(F_n)_{ij} = \omega^{ij} \qquad i, j = 0, 1, \dots, n-1$$

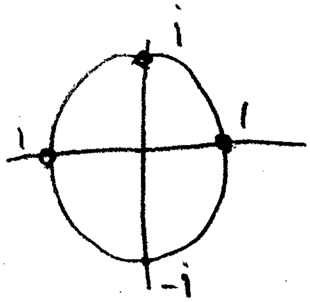


$$\omega = e^{2\pi i/n}, \qquad \omega^n = 1$$

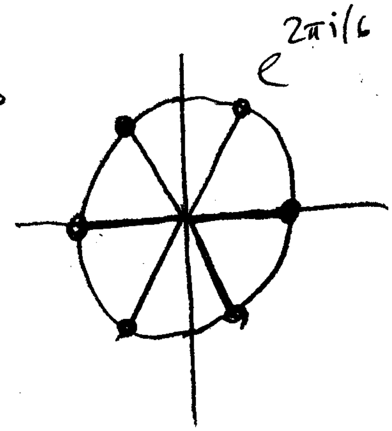
"primitive n^{th} root of unity"

<u>Note:</u> $\omega = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$

<u>Ex:</u> <u>$n = 4$</u>



<u>$n=6$</u>



$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

<u>Check:</u> Columns are <u>orthogonal</u>. $F_4^H F_4 = 4I$.

(all have length $\sqrt{n} = \sqrt{4} = 2$.)

So $\frac{1}{\sqrt{n}} F_n$ is <u>unitary</u>.

(inverse Fourier transform)

$$v \xrightarrow{\quad F_n \quad} F_n v \xrightarrow{\quad \frac{1}{n} \bar{F}_n \quad} v$$

↑ input signal / vector ↑ Fourier transform of v

Ex:
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

single impulse at time $t=0$

all frequencies in equal amounts.

Multiplying by F_n can be done Fast!

Neat fact: relationship b/w F_n and F_{2n}.

Ex:
$$\begin{bmatrix} F_{64} \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \overbrace{\begin{bmatrix} 1 & & 1 & & \cdots & & 1 \\ \hline & 1 & & 1 & & \cdots & & 1 \end{bmatrix}}^{P}$$

$(64)^2$ vs. $2(32^2) + 32$

$$P \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ \vdots \\ x_2 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} \qquad D = \begin{bmatrix} 1 & & & \\ & w & & \\ & & w^2 & \\ & & & \ddots \\ & & & & w^{n-1} \end{bmatrix}$$

Big idea: $F_{2n} V$ should take $(2n)^2 = 4n$ operations.

But we can do it in $2 \cdot n^2$ operations.

This reduces $O(n^2)$ to $O(n \log_2 n)$ in general.