

(4) Similar matrices

Def: A, B are similar if $B = M^{-1}AM$ for some M

$$\hookrightarrow \Rightarrow A = MBM^{-1}$$

Ex: A has n e-vectors. Then $S^{-1}AS = \Lambda$, so

" A is similar to Λ ."

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda = 3, 1$, $\text{tr} = 4$
 $\text{det} = 3$.

$$\begin{matrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} & = & \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} & = & \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix} \\ S^{-1} & A & S & & & B & & \end{matrix}$$

Other similar matrices: $\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$, $\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$, etc.

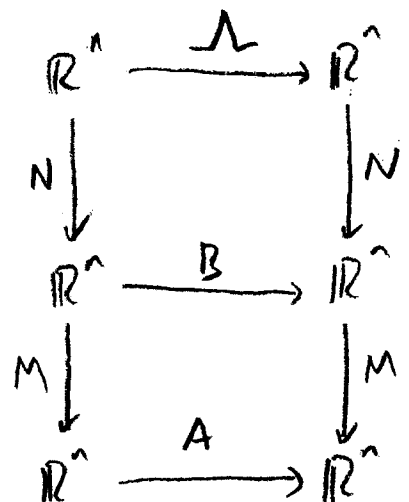
* Key idea: Similar matrices have the same e-values.

Why: $Ax = \lambda x$ ($B = M^{-1}AM$)

$$(M^{-1}AM)M^{-1}x = \lambda M^{-1}x$$

$$B(M^{-1}x) = \lambda(M^{-1}x)$$

$\Rightarrow \lambda$ is an e-value of B ; e-vector $M^{-1}x$.



(2)

* Thus, any two matrices with the same set of distinct e-values are similar (to Λ , and thus to each other).

"Bad" case: $\lambda_1 = \lambda_2$

Ex: Say $\lambda_1 = \lambda_2 = 4$. (so $\text{tr} = 8$, $\text{det} = 16$.)

One family has (only) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ $M^{-1}(4I)M = 4I M^{-1}M = 4I$.

The other family has: $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \leftarrow$ "Jordan canonical form"

Other members: $\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & 1000 \\ 0 & 4 \end{bmatrix}$.

Def: A Jordan block is a matrix $J = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$

Has only 1 e-vector

Why: $J - \lambda I = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$, $\dim N(J - \lambda I) = 1$.

Theorem (Jordan): Every square matrix A is similar to a Jordan matrix J of Jordan blocks:

$$J = \begin{bmatrix} J_1 & & & & \\ & J_2 & & & \\ & & \ddots & & \\ & & & J_d & \end{bmatrix}$$

$J_i =$ Jordan block

J is "block-diagonal."

Ex: Suppose A is a 5×5 matrix with the following

e-values: $\lambda_1 = \lambda_2 = \lambda_3 = 2, \lambda_4 = \lambda_5 = -3.$

Then A is similar to one of the following:

$$\begin{bmatrix} J_1 & & & & \\ & J_2 & & & \\ & & J_3 & & \\ & & & & \\ & & & & \end{bmatrix}$$

$$\begin{bmatrix} J_4 & & & & \\ & J_5 & & & \\ & & J_6 & & \\ & & & & \\ & & & & \end{bmatrix}$$

- Note:
- $\dim N(J_1, -2I) = 3$ 3 e-vectors for $\lambda = 2$
 - $\dim N(J_2, -2I) = 2$ 2 e-vectors for $\lambda = 2$
 - $\dim N(J_3, -2I) = 1$ 1 e-vector for $\lambda = 2$

4

Even worse case:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

vs.

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dim N(A) = 2$$

$$\dim N(B) = 2$$

Both have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, 2 e-vectors!

But A & B are not similar (by Jordan's theorem.)