4 Similar matrices

<u>Def</u>: A, B are <u>similar</u> if $\boxed{B = M^{-1}AM}$ for some M

$\Rightarrow A = MBM^{-1}$

<u>Ex</u>: A has n e-vectors. Then $S^{-1}AS = \Lambda$, so

"A is similar to Λ."

<u>Ex</u>: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda = 3, 1$, $\mathrm{tr} = 4$, $\det = 3$.

$$\underbrace{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}}_{S^{-1}} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{S} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}_{B}$$

Other similar matrices: $\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$, $\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$, etc.

* <u>Key idea</u>: Similar matrices have the same e-values.

<u>Why</u>. $Ax = \lambda x$ $\qquad (B = M^{-1}AM)$

$$(M^{-1}AM)M^{-1}x = \lambda M^{-1}x$$

$$B(M^{-1}x) = \lambda(M^{-1}x)$$

$\Rightarrow \lambda$ is an e-value of B; e-vector $M^{-1}x$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\Lambda"] \arrow[d, "N"'] & \mathbb{R}^n \arrow[d, "N"] \\ \mathbb{R}^n \arrow[r, "B"] \arrow[d, "M"'] & \mathbb{R}^n \arrow[d, "M"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \end{tikzcd}$$

(2)

* Thus, any two matrices with the same set of distinct e-values are similar (to Λ, and thus to each other).

"Bad" case: $\lambda_1 = \lambda_2$

Ex: Say $\lambda_1 = \lambda_2 = 4$. (so $tr = 8$, $\det = 16$.)

One family has (only) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ $\qquad M^{-1}(4I)M = 4I\, M^{-1}M$
$= 4I$.

The other family has: $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ ← "Jordan canonical form"

other members: $\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & 1000 \\ 0 & 4 \end{bmatrix}$.

Def: A Jordan block is a matrix $J = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$

Has only 1 e-vector

why: $J - \lambda I = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$, $\dim N(J - \lambda I) = 1$.

Theorem (Jordan): Every square matrix A is similar to a Jordan matrix J of Jordan blocks:

$$J = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_d} \end{bmatrix}$$

J_i : Jordan block

J is "block-diagonal."

<u>Ex:</u> Suppose A is a 5×5 matrix with the following e-values: $\lambda_1 = \lambda_2 = \lambda_3 = 2$, $\lambda_4 = \lambda_5 = -3$.

Then A is similar to <u>one</u> of the following:

J_1

$$\begin{bmatrix} \boxed{2} & & & & \\ & \boxed{2} & & & \\ & & \boxed{2} & & \\ & & & \boxed{-3} & \\ & & & & \boxed{-3} \end{bmatrix}$$

J_2

$$\begin{bmatrix} 2 & 1 & & & \\ & 2 & & & \\ & & \boxed{2} & & \\ & & & \boxed{-3} & \\ & & & & \boxed{-3} \end{bmatrix}$$

J_3

$$\begin{bmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & \boxed{-3} & \\ & & & & \boxed{-3} \end{bmatrix}$$

J_4

$$\begin{bmatrix} \boxed{2} & & & & \\ & \boxed{2} & & & \\ & & \boxed{2} & & \\ & & & -3 & 1 \\ & & & & -3 \end{bmatrix}$$

J_5

$$\begin{bmatrix} 2 & 1 & & & \\ & 2 & & & \\ & & \boxed{2} & & \\ & & & -3 & 1 \\ & & & & -3 \end{bmatrix}$$

J_6

$$\begin{bmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & -3 & 1 \\ & & & & -3 \end{bmatrix}$$

<u>Note:</u>

$\dim N(J_1 - 2I) = 3$ 3 e-vectors for $\lambda = 2$

$\dim N(J_1 - 2I) = 2$ 2 e-vectors for $\lambda = 2$

$\dim N(J_1 - 2I) = 1$ 1 e-vector for $\lambda = 2$

Even worse case:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & & 0 \end{bmatrix} \quad \text{vs.} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim N(A) = 2$ $\qquad\qquad$ $\dim N(B) = 2$

Both have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, 2 e-vectors!

But A & B are not similar (by Jordan's theorem.)