16 Linear transformations

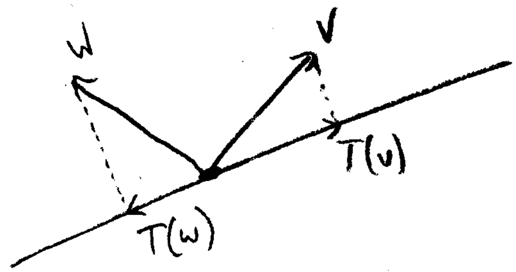
Def: A linear transformation is a function $T: V \to W$ between vector spaces satisfying

① $T(u+v) = T(u) + T(v)$

② $T(cv) = c\,T(v)$

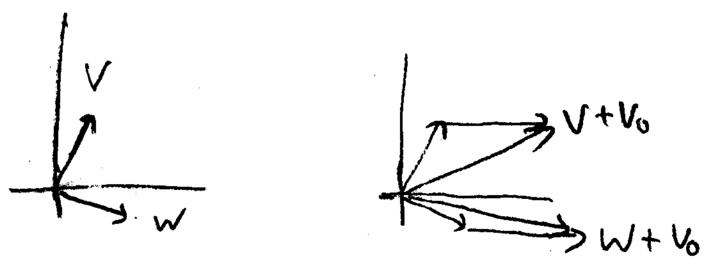
$\Big\}$ OR: $T(cu+dv) = cT(u) + dT(v)$.

Ex 1: Projection $T: \mathbb{R}^2 \to \mathbb{R}^2$



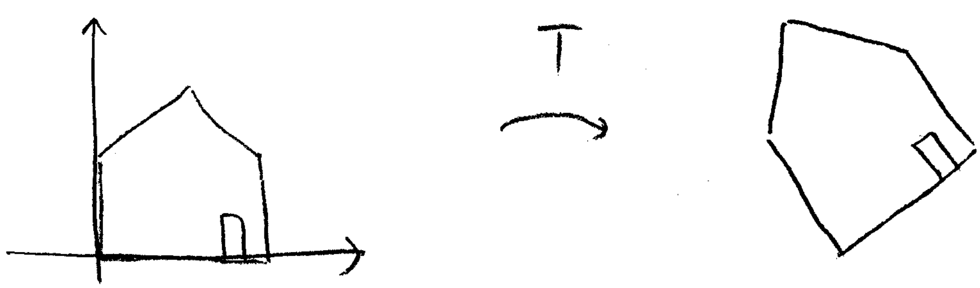
Non-ex 1: Shift by v_0

Not linear (why?)



Remark: $T(\vec{0}) = T(2\cdot\vec{0}) = 2\,T(\vec{0}) \Rightarrow T(\vec{0}) = \vec{0}$.

Ex 2: Rotation by $45°$



Non-ex 2: $T(v) = \|v\|$ (say $T: \mathbb{R}^3 \to \mathbb{R}^3$).

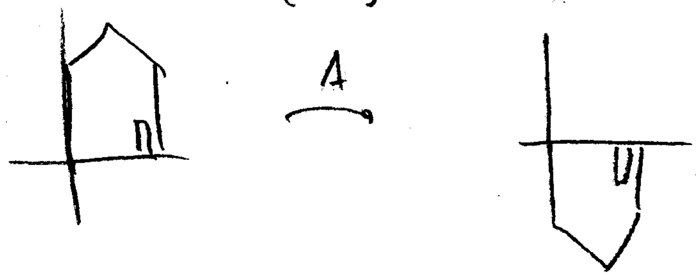
Why: $T(-2v) \neq -2\,T(v) = -2\,\|v\|$.

Ex 3: $T(v) = Av$ ← matrix

linear: $A(v+w) = Av + Aw$ ✓

$A(cv) = cAv$. ✓

If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,



Goal: Understand linear transformations.

How: Find the matrix that lies behind (need a basis!)

Defining a basis leads to coordinates:

e.g., $v = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ Basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Coordinates

Remark: Knowing what $T: V \to W$ does on a basis $\{v_1, \ldots, v_n\}$ determines everything!

e.g., if $v = c_1 v_1 + \cdots + c_n v_n$

then $T(v) = c_1 T(v_1) + \cdots + c_n T(v_n)$.

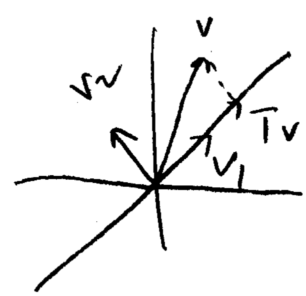
How to construct a matrix A that represents $T: \mathbb{R}^n \to \mathbb{R}^m$?

- Chose an input basis $v_1, \dots, v_n$ for $\mathbb{R}^n$
- Choose an output basis $w_1, \dots, w_m$ for $\mathbb{R}^m$.

Write
$$T(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$$
$$T(v_2) = a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m$$
$$\vdots$$

Then
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$
$$\uparrow T(v_1) \quad \uparrow T(v_2) \quad \uparrow T(v_n)$$

<u>Ex</u> Projection on $45°$ line:



<u>Standard</u> basis: e_1, e_2 (input & output)

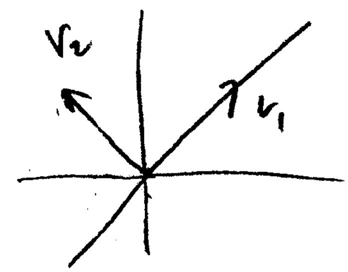
$$P = \frac{aa^T}{a^Ta} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

<u>Eigenvector</u> basis v_1 (on line), v_2 ($\perp$ to line)

(input & output)

If $v = c_1 v_1 + c_2 v_2$

then $T(v) = c_1 v_1$

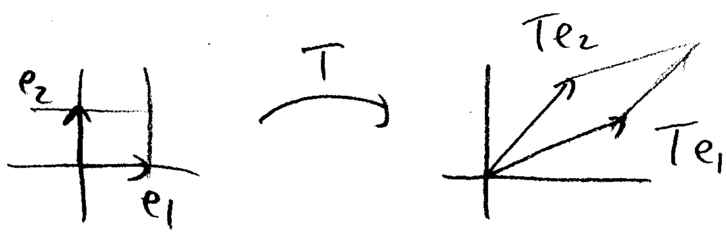


Note: $T(v_1) = 1\, v_1 + 0\, v_2$
$T(v_2) = 0\, v_1 + 1\, v_2$ $\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$

i.e., $(c_1, c_2) \xmapsto{A} (c_1, c_2)$ in these coordinates.

Ex: If T is invertible, then if we pick the input & output basis just right, the matrix is I!

Here:



input basis: $v_1 = e_1$
$v_2 = e_2$

output basis: $w_1 = Te_1$
$w_2 = Te_2$

If $v = c_1 e_1 + c_2 e_2 = (c_1, c_2)$ wrt input basis

$T(v) = c_1 T(e_1) + c_2 T(e_2) = c_1 w_1 + c_2 w_2 = (c_1, c_2)$
wrt output basis

$T(e_1) = 1\, w_1 + 0\, w_2$
$T(e_2) = 0\, w_1 + 1\, w_2$ $\Rightarrow$ matrix is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

Ex: $T = \frac{d}{dx}$ Input $c_0 + c_1 x + c_2 x^2$ basis: $1, x, x^2$

Output $c_1 x + 2c_2 x$ basis $1, x$

$$T(1) = \boxed{0} \cdot 1 + \boxed{0} \cdot x$$
$$T(x) = \boxed{1} \cdot 1 + \boxed{0} \cdot x$$
$$T(x^2) = \boxed{1} \cdot 1 + \boxed{2} \cdot x$$

$$A = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \frac{d}{dx}1 & \frac{d}{dx}x & \frac{d}{dx}x^2 \end{array} \end{array}$$